Gauss' Law
Part 2

Review:

Last time we talked about Gauss' law. Simply stated, it is that the total electric flux through any surface is equal to the charge enclosed by the surface, divided by \( \varepsilon_0 \).

\[
\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}
\]

Where electric flux is a measure of the electric field lines through a surface and is analogous to fluid flow.

(But just an analogy).

This is useful because it helps us calculate the electric field since we can choose whatever surface makes the calculation easiest.

For example, an infinitely long line of charge with charge density \( \lambda \) \( \vec{E} \) is perpendicular to the line.

\[
\oint \vec{E} \cdot d\vec{A} = \int_{\text{top}}^{\text{bottom}} \vec{E} \cdot d\vec{A} = \int E \, dA = E \, \pi r^2 L
\]

\[
= \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{\pi \lambda L}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{\pi r^2 \varepsilon_0}
\]
We worked several examples, and will do more later today.

But first, a reminder of the most important result from Gauss’ law.

There is no $E$ field inside a closed, hollow conductor. Know why.

If we have charge on a solid conductor, it must lie on the surface, because any gaussian surface we imagine inside the conductor must have $E=0$ on it so $\int E \cdot dA = 0$. No enclosed charge. ($E=0$ or charge would move.)

We can imagine the surface just inside the conductor’s surface $\Rightarrow$ all charge must be on outer surface.
Suppose now that there is a hollow in the conductor, $\mathbf{E} = 0$ in the conducting shell, so $\text{gauss} = 0$.

But what if I put a charge in there?! $\mathbf{E}$ still must be zero in the conductor, so $\text{gauss} = 0$.

What gives, eh?!

Well, $\text{gauss}$ is the net charge enclosed. $\text{Net} = 0$, because an equal but opposite (positive) charge is induced on the inside surface of the hollow, so net = 0.

I can't cancel charge though, so there must be an equal + charge on the outside of the conductor. (Just like the "external" induction on a sphere.

Finally, if we imagine a gaussian surface around the outside of the conductor, its total $\text{gauss} = \text{the} \ + \ \text{charge} \ \text{in the hollow}$. 
Interestingly, if the inner \( \Theta \) charge were offset, then the charge on the inner surface is non-uniform, but the charge on the outer surface is uniform. It must be uniform because \( \vec{E} = 0 \) in the conductor.

\[ \Rightarrow \] A spherical conduction shell makes an off-center charge inside it look centered.

**DEMO** zero \( q \) in conducting shell using oscilloscope.

So, now let's do more examples to gain familiarity.
An infinite sheet of charge, w/ charge density $\sigma$.

$$E = \frac{\sigma}{2\varepsilon_0}$$

Know how to get this.

How about

Same but opposite

How about a conducting sheet.

$E=0$ inside so

$$E = \frac{\sigma}{\varepsilon_0}$$

Suppose we took a + charged and - charged non-conducting sheets and put them together.

Electric fields add.

$\Rightarrow$ Inside $E = 2 \times \frac{\sigma}{\varepsilon_0}$.

Outside $E = 0$. 
Sample problem:

Three, infinite non-conducting sheets with charge densities \( \sigma, 2\sigma, 3\sigma \):

What is \( E \) at all points?

\[
\begin{align*}
\frac{3\sigma}{2\varepsilon_0} & \quad \frac{3\sigma}{2\varepsilon_0} \rightarrow \quad \frac{3\sigma}{2\varepsilon_0} \rightarrow \\
& \quad \frac{0}{2\varepsilon_0} \quad \frac{0}{2\varepsilon_0} \rightarrow \quad \frac{0}{2\varepsilon_0} \rightarrow \\
& \quad \frac{2\sigma}{2\varepsilon_0} \quad \frac{2\sigma}{2\varepsilon_0} \rightarrow \quad \frac{2\sigma}{2\varepsilon_0} \rightarrow \\
\frac{3\sigma}{\varepsilon_0} = 0 & \quad \frac{0}{\varepsilon_0} \rightarrow \frac{3\sigma}{\varepsilon_0} \rightarrow
\end{align*}
\]

The key idea is that electric field add, as vectors.
Sample Problem:
I have an infinite sheet of charge density \( \sigma \) from which I cut a disk of radius \( R \). What is \( E \) above the resulting hole?

It is \( \frac{\sigma}{2\varepsilon_0} \) from the missing disk.

Messy but calculable. See example 21.12.

What must we get in the limiting cases?

Sample problem: What are the units of \( E_0 \)?

\[ E \cdot A = \frac{q}{E_0} \quad \Rightarrow \quad E_0 = \frac{q}{E \cdot A} = \frac{C}{N \cdot m^2} = \frac{C^2}{N m^2} \]
Sample Problem:

What is \( \vec{E} \) for a spherical, uniform charge distribution with charge density \( \rho \)?

This is sort of like a planet, but not \( \vec{F} \).

\[
\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}
\]

\[
E(r) = \frac{\rho r}{3 \varepsilon_0}
\]

Outside:

\[
E = \frac{\rho R^3}{3 \varepsilon_0}
\]

Note, that \( \rho = \frac{Q}{\frac{4}{3} \pi R^3} \), so

\[
\vec{E}(r) = \begin{cases} 
\frac{Q r^2}{4 \pi \varepsilon_0 R^3} & r \leq R \\
\frac{Q}{4 \pi \varepsilon_0 r^2} & r > R 
\end{cases}
\]
Sample Problem:

Sphere of radius $a$ with charge $+q$ (uniformly distributed) sits inside a hollow sphere of inner radius $b$, outer radius $c$, and charge $-q$.

(a) What is $\hat{E}(r)$ inside the sphere, $r < a$?

$$E = \frac{q}{4\pi \varepsilon_0 r^2}$$

(b) What is $\hat{E}(r)$ for $a < r < b$?

$$E = \frac{q}{4\pi \varepsilon_0 r^2}$$

(c) What is $\hat{E}(r)$ for $b < r < c$? Zero. Conductor.

(d) What is $\hat{E}(r)$ for $r > c$? Zero. No net charge.

(e) What charge is on the inner surface of the shell? -q (induced).

(f) What charge is on the outer surface of the shell? Zero. -q and +q induced.

(g) What if the outer shell were not symmetric? Then $\Phi_E = 0$ and $\varepsilon_0$ still holds, but we can't find a simple Gaussian surface.
Sample problem:

A nonconducting ball of mass \( m \) and charge \( q \) hangs from a thread near a large, charged nonconducting sheet. What is the charge density of the sheet?

\[ F_E = qE = q \cdot \frac{\sigma}{2\varepsilon_0} \]

\[ mg \]

\[ T \cos \theta = mg \Rightarrow T = \frac{mg}{\cos \theta} \]

\[ T \sin \theta = q \cdot \frac{\sigma}{2\varepsilon_0} \]

\[ \Rightarrow mg \tan \theta = \frac{q \cdot \sigma}{2\varepsilon_0} \]

\[ \therefore \sigma = \frac{2\varepsilon_0 mg \tan \theta}{q} \]
Sample Problem:

If an atom is modelled as a tiny positive nucleus with charge \(Ze\) and a uniform cloud of electrons with charge \(-Ze\) spread over a sphere of radius \(R\). What is \(E(r)\)?

\[
E(r) = E_{\text{nucleus}} + E_{\text{electrons}} = 0 \text{ outside } R
\]

**Inside** \(r \leq R\):

\[
E_{\text{nucleus}} = \frac{Ze}{4\pi\varepsilon_0 r^2}
\]

**Total**:

\[
E(r) = \frac{-Ze}{4\pi R^3} \left(\frac{1}{r^2} - \frac{r}{R^3}\right) \quad r \leq R
\]
Sample Problem:
Suppose we have a non-conducting sphere with a non-uniform charge distribution given by \( \rho(r) = \rho_0 \frac{r}{R} \), where \( R \) is the radius of the sphere and \( \rho_0 \) is some constant.

a) What is the Electric Field inside the sphere?

\[ \mathbf{E} \text{ is radial by symmetry, so use a Gaussian surface that is a sphere.} \]

\[ \oint \mathbf{E} \cdot d\mathbf{A} = E(r) 4\pi r^2 = \frac{\partial \mathbf{E}}{\partial r} = \frac{Q(r)}{\varepsilon_0} = \frac{Q(r)}{\varepsilon_0} \]

\[ Q(r) = \int_0^r \rho(r') dV \int_0^r \frac{\rho_0 r'}{R} 4\pi r' r'^2 dr' \]

\[ = \frac{4\pi \rho_0 \frac{r^4}{4}}{R} = \frac{\pi \rho_0 \frac{r^4}{4}}{R} \]

So,

\[ E(r) = \frac{Q(r)}{\varepsilon_0} \frac{1}{4\pi r^2} = \frac{\pi \rho_0 \frac{r^4}{4}}{4\pi r^2 \varepsilon_0 R} \]

\[ \mathbf{E}(r) = \frac{\rho_0 \hat{r}}{4\varepsilon_0 R} \]
Sample Problem:

Two non-conducting spheres of radius $R$ have charges $q_1$ and $q_2$ uniformly distributed on them. What must be the ratio of their charges, $q_2/q_1$, to make $E=0$ at a point $R/2$ from the center of sphere 1 when they are touching as below.

![Diagram of two spheres touching]

The electric field at the point is just the sum of the two fields $E_1$ and $E_2$ from the two spheres.

$$E_1(R/2) = 4\pi (\frac{R}{2})^2 = \frac{q_1}{\frac{4}{3}\pi R^3} / \varepsilon_0 = \frac{q_1}{8\varepsilon_0}$$

$$E_2(R+R/2) = 4\pi (\frac{3R}{2})^2 = \frac{q_2}{\varepsilon_0}$$

$$\Rightarrow E_1 = \frac{q_1}{8\pi\varepsilon_0 R^2} \hat{x} \frac{1}{3} = E_2 = \frac{q_2}{9\pi\varepsilon_0 R^2} (-x)$$

$$E_1 + E_2 = 0 \Rightarrow \frac{q_1}{8\pi\varepsilon_0 R^2} = \frac{q_2}{9\pi\varepsilon_0 R^2}$$

$$\frac{q_2}{q_1} = 9$$

(Note that the shell of sphere 1, outside of $R/2$, cancels only because it is symmetric.)
A small charged ball is inside a conducting spherical shell of radius R. What charge is on the inner and outer surfaces of the shell if the ball and shell have net charges of:

a) \(+4q, 0\)

b) \(-6q, +10q\)

c) \(+16q, -12q\)

a) \(E\) must be 0 inside the conducting shell, so 
\(Q_{\text{enclosed}} = 0 \Rightarrow -4q\) on inner surface.
But net = 0 charge \(\Rightarrow +4q\) on outer surface.

b) \(E = 0\) inside conductor \(\Rightarrow Q_{\text{enclosed}} = 0 \Rightarrow +6q\) on inner surf.
Net charge of \(+10q\) \(\Rightarrow +4q\) on outer surface.

\(c\), \(-16q\) on inner surface.
\(+4q\) on outer surface.

What is \(\vec{E}\) at some point \(|\vec{r}| > R\)?
Sample question:

A central metal ball is surrounded by two spherical metal shells, that are concentric with the ball. They are charged with $Q$, $3Q$, and $5Q$, respectively.

Imagine three spherical gaussian surrounding them, radii $R$, $2R$, $3R$

Rank the gaussian surfaces according to the magnitude of $E$ at any point on that surface.

Well, $EA = \frac{Q}{\epsilon_0}$. $E = \frac{\frac{Q}{\epsilon_0}}{4\pi(2R)^2}$.

$\Rightarrow E_1 = \frac{Q}{4\pi\epsilon_0 R^2}$

$E_2 = \frac{4Q}{4\pi\epsilon_0 (2R)^2} = E_1$

$E_3 = \frac{9Q}{4\pi\epsilon_0 (3R)^2} = E_1$

$\Rightarrow$ All equal.
What is \( E \) at a point away from a ring of charge?

- Can't use Gauss' law because there is no simplifying symmetry. Must integrate.

What is \( E \) outside a cube of charge?

Again, no global symmetry. Must integrate. But there is a special case, where Gauss' law is useful near the faces where it is approximately infinite.