Lorentz Transformations

Now we can replace the Galilean transformations \((x' \rightarrow x)\) with relativistic transformations. (They are called Lorentz transformations since Lorentz came up with them before Einstein, he just didn't make the bold claim that they represented space-time itself.)

\[ x(t) = x' + ut \] becomes \[ x(t) = ut + \frac{x'}{\gamma} \]

contracted

\[ x'(t) = \gamma(x - ut) \]

\[ y'(t) = y(t) \quad \text{and} \quad z'(t) = z(t) \]

How can we connect \(t\) and \(t'\)?

By symmetry, if we switch \(S\) and \(S'\) then \[ x' \leftrightarrow x \quad t' \leftrightarrow t \] and \(u \rightarrow -u\).

So,

\[ x = ut + \frac{x'}{\gamma} \quad \Rightarrow \quad x' = -ut' + \frac{x}{\gamma} \]

\[ x' = \gamma(x - ut) = -ut' + \frac{x}{\gamma} \]

\[ t' = \gamma \left( x - ut \right) - \frac{x}{\gamma} \]

\[ -u \]
\[ t' = \delta t + x \left( \frac{\sqrt{1-u^2/c^2}}{u} - \frac{1}{u \sqrt{1-u^2/c^2}} \right) \]

\[ = \delta t + x \left( \frac{1-u^2/c^2}{u \sqrt{1-u^2/c^2}} - \frac{1}{u \sqrt{1-u^2/c^2}} \right) \]

\[ = \delta t - \frac{x u/c^2}{\sqrt{1-u^2/c^2}} \]

\[ t' = \chi \left( t - \frac{x u}{c^2} \right) \]

Note that space and time are now mixed together in the transformation.

We also want to expand this to transform the other variables we always use in physics. \( \chi, p, E, \) etc.

\[ V_x' = \frac{dx'}{dt'} \quad dx' = \chi \, dx - \chi \, u \, dt \]

\[ dt' = \chi \, dt - \frac{\chi \, u \, dx}{c^2} \]

\[ V_x' = \frac{\chi \, dx - \chi \, u \, dt}{\chi \, dt - \frac{\chi \, u \, dx}{c^2}} \cdot \frac{1}{\chi \, dt} \]

\[ = \frac{dx}{dt} - \frac{u}{1 - \frac{V_x u}{c^2}} = \frac{V_x - u}{1 - \frac{V_x u}{c^2}} \]
Again, we can use symmetry to determine

\[ v_x(t) = \frac{v_x' + u}{1 + uv_x/c^2} \]

just changing \( v_x \leftrightarrow v_x' \) and \( u \rightarrow -u \).

Note that these transformation equations were derived assuming the two reference frames move relative to each other along their common \( x \)-axes. If the motion is along a different axis then these equations don’t apply.

Also, we defined \( t=0 \) and \( t'=0 \) when the origins of the two frames coincide. If otherwise, equations don’t apply.

As usual, “it is not about memorizing the equations, “it” is about understanding them.”
So, what about the other "kinematic" observables like momentum, force, and energy.

To address this, it is easiest to imagine a reference frame sitting on top of a particle. In that frame, everything is zero.

\[ (x, y, z) = 0 \quad \dot{v} = 0, \text{ time just ticks by.} \]

What is its momentum? Zero.
What is its energy? Well if \( v = 0 \), the \( \frac{1}{2}mv^2 = 0 \), but there is more to this story.

This becomes clear when we imagine a simple reaction. If the object splits into two pieces (imagine a spring pushed them apart), then

Initially \[ m \quad 0 \quad 0 \]

After \[ \frac{mv}{w} \quad 0 \quad \frac{mv}{w} \]

\[ \mathbf{P}_1 = 0 \quad \sum \mathbf{P}_i = \mathbf{v} + (-\mathbf{v}) = 0 \]

Good, our old favorite conservation of momentum.
But all the laws of physics are supposed to hold in all reference frames, so let's check if momentum is conserved in a frame that is moving relative to this rest frame, at speed \( u \). (Imagine the rest frame is moving past us at \( u \), so we are \( S \) and it is \( S' \).)

In \( S \):

Initially:

\[
\Sigma P_i = m_u + m_u = 2m_u
\]

After:

\[
\Sigma P_i = m_v, + m_v = ?
\]

We need to use the Lorentz x-form to calculate \( v, \) and \( v' \) in our frame.

Final momentum:

\[
\Sigma P_i = m \left( \frac{v + u}{1 + uv/c^2} \right) + m \left( \frac{-v + u}{1 - uv/c^2} \right)
\]

\[
= \frac{m}{(1 + uv/c^2)(1 - uv/c^2)} \left[ (v + u)(1 - uv/c^2) + (u - v)(1 + uv/c^2) \right]
\]

\[
= \frac{m}{1 - u^2v^2/c^4} \left[ v + u + u - v - uv^2 - u^2v + u^2v - u^2v - uv^2 \right]
\]

\[
= m \left[ au - auv^2/c^2 \right] = 2m_u \left[ \frac{1 - \frac{v^2}{c^2}}{1 - \frac{u^2v^2}{c^4}} \right]
\]
So, 
\[ P_f^{\text{Total}} = P_i^{\text{Total}} \Rightarrow \text{Momentum is not conserved!} \]
as viewed in the S reference frame, even when we apply the Lorentz transformations.

That violates the first postulate, that all laws of physics apply in all inertial reference frames.

*Something is wrong!*

\[ amu \neq amu \left[ \frac{1 - \frac{v^2}{c^2}}{1 - \frac{u^2}{c^2}} \right] = amu \sqrt{1 - \frac{v^2}{c^2}} \]

because this is not the proper description of momentum. If instead, we define momentum as

\[ \vec{p} = \frac{mv}{\sqrt{1 - \left( \frac{v}{c} \right)^2}} \]

then we will find that momentum is conserved in all frames.

*Note that I wrote v, not \( u \), in the denominator. Why?*
We used \( u \) as the speed of one frame relative to another. In the above example, \( v \) and \( u \) were needed because things were moving within frames.

I use only \( v \) in
\[
\vec{p} = m \vec{v} / \sqrt{1 - \frac{v^2}{c^2}}
\]
because we can always think of an object in its rest frame and determine its velocity in any other frame. So \( v \) is its speed in your frame.

So,
\[
\vec{p} = m \vec{v}.
\]

And
\[
\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt} \left( \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \frac{ma}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}
\]
(see next page for derivation)

Find that if \( F \) is constant, \( a \) decreases with \( v \),
\[
a = \frac{F}{m} \left(1 - \frac{v^2}{c^2}\right)^{3/2}
\]
and goes to zero as \( v \rightarrow c \). Would need \( \infty F \) to surpass.
Now, what about energy?

\[ W = \int F dx = \int \frac{dp}{dt} dx \]

\[
\frac{dp}{dt} = \frac{d}{dt} \left( \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \frac{mv}{(1 - \frac{v^2}{c^2})^{3/2}} \left( -\frac{1}{2} \right) (a) + \]

\[ + \frac{ma}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ = ma \left[ \frac{v^2}{(1 - \frac{v^2}{c^2})^{3/2}} + \frac{1 - \frac{v^2}{c^2}}{(1 - \frac{v^2}{c^2})^{3/2}} \right] \]

\[ = \frac{ma}{(1 - \frac{v^2}{c^2})^{3/2}} \]

\[ \Rightarrow W = \int \frac{m \, dv}{dt} \, dx = v \, dt \]

\[ = \int \frac{m \, v \, dv}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \]

accelerated from 0 to final speed \( u \).

\[ = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} - mc^2 \]

\[ = mc^2 \left( \gamma - 1 \right) \]
By the work-energy theorem,

\[ K = W = (\gamma - 1) mc^2 = 8mc^2 - mc^2 \]

The \( mc^2 \) term doesn't depend on speed. It is the "rest energy".

So, the total energy,

\[ E = K + mc^2 = \gamma mc^2 \]

Note, also, that since \( p = \gamma mv \)

\[
\begin{align*}
 p^2c^2 + (mc^2)^2 &= \gamma^2 m^2 v^2c^2 + m^2 c^4 \\
 &= m^2 \left[ \frac{v^2c^2}{1 - \frac{v^2}{c^2}} + \frac{c^4 - v^2c^2}{1 - \frac{v^2}{c^2}} \right] \\
 &= m^2 \left[ \frac{c^4}{1 - \frac{v^2}{c^2}} \right] \\
 &= \frac{m^2c^4}{1 - \frac{v^2}{c^2}} = \left[ \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \right]^2 \\
 &= \left[ \gamma mc^2 \right]^2 = E^2
\end{align*}
\]

So, \( E^2 = p^2c^2 + (mc^2)^2 \)
or

\[ E = \sqrt{p^2c^2 + m^2c^4} \]

If \( p=0 \), then \( E=mc^2 \) ... the famous eqn. That is just the "rest mass" energy.

If \( m=0 \), then

\[ E = pc \]

So, we can calculate the Energy vs. momentum for a (massless) photon this way.