Last time we briefly surveyed the state of physics as of ~1900 and began to discuss relativity.

Relativity describes how we transform between reference frames that are in relative motion.

We are considering the special case of inertial reference frames, that is ones moving at constant relative velocity.

or Galileu's

In Newton's "old fashioned" (or classical) way, the transformation from one frame to another was

\[
\begin{align*}
x(t) &= x'(t) + ut \\
y(t) &= y'(t) \\
z(t) &= z'(t)
\end{align*}
\]

where the \( S' \) frame is moving at speed \( u \) along \( x \) relative to frame \( S \).

\[
\frac{dx}{dt} = \frac{dx'}{dt} + u = V_x'(t) + u
\]

and similarly for \( a_x(t) \).

Newton's laws apply equally in all such inertial reference frames.
The postulates of Einstein’s special relativity differ from the classical relativity. They are:

1. All the laws of physics apply equally in all inertial reference frames.

2. Light propagates at a speed \( c \), in vacuum, when measured in any inertial reference frame.

We saw the first consequence of that by imagining a beam of light bouncing off a mirror.

In the “rest frame”

\[
\Delta t_0 = \frac{2d}{c}
\]

In a moving frame

\[
\Delta t = \frac{2d + \left(\frac{1}{\gamma^2}\right)}{c}
\]

where we use \( \frac{1}{\gamma} \) in both cases because of the second postulate.

Solving for \( \Delta t \) we get

\[
\Delta t = \frac{\Delta t_0}{\sqrt{1 - \left(\frac{\nu}{c}\right)^2}}
\]

⇒ Time dilation
This time dilation means that observers in different frames have different measures of time.

You, stationary, watching me go past will observe my watch to be running slow. I won't observe anything wrong with my watch, indeed I'd observe your watch to be slow!

This is a critical difference between Galilean and Einstein's relativity. The old

$$x(t) = x'(t') + ut \Rightarrow \frac{dx(t)}{dt} = \frac{dx'(t')}{dt} + u$$

no longer works because $t' \neq t$.

So, we now need to very carefully check our assumptions (intuition) about time.

So, we need to expand our coordinate system to be not just spatial coordinates, but also time $\Rightarrow$ space-time coordinates.

That isn't really new, we already include time in discussing (defining) an event: where and when.
Now let's make that specific. I have a coordinate system, \(x, y, z\), to which I'll add a clock, \(t\).

Any event can be specified by a point in this 4D space \((t, x, y, z)\). I put \(t\) in the "0th" element and left space in their previous positions as 1, 2, 3.

Just as we could specify a position as a vector \(\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}\) from the origin of the coordinate system to the point, we can likewise specify an event with a 4-vector. Time is not as "directional" but it is a useful concept later for things like momentum \(p = (E, \vec{p})\).

We won't actually use this in our treatment, but I thought it worth mentioning the "jargon."

One complication of using time as a coordinate is "how do I measure the time at \((x, y, z)\)? Since it takes time for light to travel from \((x, y, z)\) to my eye — wherever that is, say the origin, I would have to unfold that time difference.

We'll avoid that nuisance by imagining that synchronized clocks are placed at every point in space, so we can measure the time of an event, at the location of the event.
How we arrange this isn't important--or relevant. Just keep in mind that the time you see an event isn't what it means, it is when it happened.

To be explicit, let's imagine two flash bulbs that flash simultaneously, on opposite ends of the room. You could arrange this with synchronized clocks.

In a stationary frame, someone at the center will observe the flashes as arriving simultaneously.
\[ \Delta t_1 = \frac{d_1}{c}, \quad \Delta t_2 = \frac{d_2}{c}, \quad d_1 = d_2 \Rightarrow \Delta t_1 = \Delta t_2. \]

But someone in a moving frame will make a different observation.

Suppose I am moving, in my frame \( S' \), at a speed \( u \) along a common \( x \)-axis. So I am moving toward flash 2 and away from flash 1. So, flash 2 reaches me first, before flash 1.

Is that because the flashes occurred at different times or when I was closer? To tell for sure we need to be careful... position and time are easily mixed here.

Two identical spaceships, one moving right at relative speed \( u \).

Observers A \& B sitting in the middle, asteroids hit both the front \& back of the ships, scraping both ships as they scrape between, and giving off a flash of light.
If observer B sees the flashes simultaneously then they must have each hit simultaneously since he measures equal distance to both.

Observer A measures equal distances to the scrutinies, but sees the right hand flash first. So he concludes that that asteroid hit first -- that is what he says that they did not hit simultaneously.

A and B disagree. Who is right? Both are.

The second postulate requires that events that are simultaneous in one reference frame need not be simultaneous in other reference frames. That seems non-intuitive, but we just have to get over that.

It is like changing frequency from Doppler shift. (Listeners in different frames hear different frequencies... that seems intuitive because we hear that all the time.)

To see that this isn't just due to light propagation speed, we can switch it. A is at rest and thinks it is simultaneous. B moves left at \( u \) and doesn't think they are simultaneous. By symmetry. The key is that neither can tell because light propagates at \( c \) for both observers.
Now, we can return to time dilation,

$$\Delta t = \Delta t_0 \sqrt{1 - \left(\frac{v}{c}\right)^2} \Rightarrow \Delta t \text{ always } > \Delta t_0.$$ 

Doesn't that make it special? Yes, because it is the one where both events (shoot laser and receive reflected pulse) occur at the same point.

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Similarly, the frame that sees equidistant events as simultaneous receives them at the same time.
Relativity of length

We saw that the time coordinate is relative.
Now, let’s see how length is also relative.

\( \Delta t \) is difference between two events

\( \Delta x \) is difference between two points on x-axis.

So, if I have an object and want to measure its length, just lay it along the x-axis’s and

\[ L = \Delta x. \]

If the rod were moving, then I could still measure its length with \( \Delta x \), but would need to measure the two x’s simultaneously.

Since simultaneity is relative we shouldn’t be surprised to find that length is also relative.

If a rod is stationary in your rest frame, and you measure it to be \( L_0 \) (subscript because in rest frame). What would I measure in my moving frame (parallel to rod at speed \( u \))?
It takes me a time $\Delta t = \frac{L_0}{u}$ to pass from one end of the rod to the other (or so says you in your frame).

This is the time between two events. $\Rightarrow L_0 = u \Delta t$

From my point of view, the same events occur at one point (me) and are thus a proper time, $\Delta t_0$, measured with a single stationary clock. So, I say

$$L = \Delta t_0 \cdot u$$

Combining our measurements,

$$L_0 = u \Delta t \quad \& \quad L = \Delta t_0 \cdot u$$

with time dilation $\Delta t = \Delta t_0 / \sqrt{1 - \frac{u^2}{c^2}}$

$$\Rightarrow \frac{L}{L_0} = \frac{\Delta t_0 \cdot u}{\Delta t + u} = \frac{\Delta t_0}{\Delta t_0 / \sqrt{1 - \frac{u^2}{c^2}}} = \sqrt{1 - \frac{u^2}{c^2}}$$

$$\Rightarrow L = L_0 \sqrt{1 - \frac{u^2}{c^2}}$$

less than one, so I say it is shorter than you do!

$\Rightarrow$ Length contraction
I am lazy and tire of writing this

$$\sqrt{1 - \frac{u^2}{c^2}}$$

over and over. Since it keeps popping up, it is worth some shorthand. So define

$$\beta = \frac{u}{c} \quad \text{Note that } \beta \leq 1$$

So,

$$l = l_0 \sqrt{1 - \beta^2}$$

To make it even shorter, let's define

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad \text{Note that } \gamma \geq 1$$

So

$$\Delta t = \gamma \Delta t_0 \quad \text{and} \quad l = l_0 / \gamma$$

A moving observer sees slower time (larger $\Delta t$) and shorter distances.
It is worth philosophizing about this a bit...

Does a moving object really shrink?

Its length is the distance from one end to the other. That distance is based on measurements.

⇒ Reality is just what we measure. We measure a shorter length so it is really shorter.

When you hear a siren coming toward you, is its frequency really f? Yes, and later, when it has passed, it is really a different f.

All that is new is a realization that reality is relative.

Someone in the ambulance understands that you hear a different f than he does.

You understand that I measure a different length, and you see why -- because from your point of view I measured the two ends at different times.
Note that this length contraction occurs only along the direction of motion, not perpendicular to it. (See the textbook for the argument).

Note also that if I have a stick of length \( L \), you will see it as shorter when I go by just as I see yours is shorter. That symmetry is required since we cannot say who is really moving.

Question: If you are packing for an interstellar flight at 0.99 c, do you need to take smaller clothes? Should the spaceship have a smaller bed?

See examples 37.4 and 37.5 in Young's Freedman. They are good exercise, both numerically & conceptually.
Lorentz Transformations

Now we can replace the Galilean transformations \((x' \rightarrow x)\) with relativistic transformations.

(They are called Lorentz transformations since Lorentz came up with them before Einstein, he just did make the bold claim that they represented space-time itself).

\[ x(t) = x' + ut \] becomes \[ x(t) = ut + \frac{x'}{\gamma} \] contracted

\[ \Rightarrow x'(t) = \gamma (x - ut) \]

\[ y'(t) = y(t) \quad \text{and} \quad z'(t) = z(t) \]

How can we connect \(t\) and \(t'\)?

By symmetry, if we switch \(S\) and \(S'\) then

\[ x' \leftrightarrow x \quad t' \leftrightarrow t \quad \text{and} \quad u \leftrightarrow -u. \]

So,

\[ x = ut + \frac{x'}{\gamma} \quad \Rightarrow \quad x' = -ut' + \frac{x}{\gamma} \]

\[ x' = \gamma (x - ut) = -ut' + \frac{x}{\gamma} \]

\[ t' = \gamma \left[ x - ut \right] - \frac{x}{\gamma} \]

\[ t' = \frac{\gamma x - ux}{-u} \]
\[ t' = \delta t + x \left( \frac{\sqrt{1-u^2/c^2}}{u} - \frac{1}{u \sqrt{1-u^2/c^2}} \right) \]

\[ = \delta t + x \left( \frac{1-u^2/c^2}{u \sqrt{1-u^2/c^2}} - \frac{1}{u \sqrt{1-u^2/c^2}} \right) \]

\[ = \delta t - \frac{xu/c^2}{\sqrt{1-u^2/c^2}} \]

\[ t' = \gamma \left( t - \frac{xu}{c^2} \right) \]

Note that space and time are now mixed together in the transformation.

We also want to expand this to transform the other variables we always use in physics. \( v, p, E, \text{ etc.} \)

\[ v'_x = \frac{dx'}{dt'} \quad dx' = \gamma dx - \gamma u dt \]

\[ \frac{dx'}{dt'} = \gamma \frac{dt}{dt} - \gamma u_c \frac{dx}{dx} \]

\[ = \frac{dx}{dt} - u = \frac{v_x - u}{1 - \frac{u \cdot u}{c^2}} \]