Magnetic Force 2

Last time we worked an example of the magnetic force on a moving charged particle.

\[ \vec{F} = q \vec{v} \times \vec{B} \]

And we saw demos of the magnetic force on a current carrying wire. That is just the net (summed) force from many moving charges.

Now, I want to derive the formula for the force on a current carrying wire.

Don't let the word "derive" bore or distract you. You might want me to just state the formula---as was probably done in high school physics. But to understand the physics we need to understand where the formula comes from.

If we have a straight wire of length \( l \) carrying a current, \( I \), that is immersed in a magnetic field, perpendicular to \( l \), then \( \vec{F} \) is just the sum of the forces from the individual charges.
\[ \vec{F} = \sum_i q \vec{v}_i \times \vec{B} \rightarrow \int dq \vec{v}_i \times \vec{B} \]

Integrate over the whole wire.

I don’t want to count every electron, so let’s recall the macroscopic description of current.

\[ I = \frac{dQ}{dt} = \frac{\text{charge moving past some point}}{\text{an infinitesimal time}} \]

\[ = n q v_d A \quad (n = \text{charge carrier density}) \]

because \( n q v_d A dt \) is \( \frac{\text{C}}{\text{vol}} \cdot \text{Vol.} \)

\( V_d \) is the slow, effective drift velocity.

Each individual electron flies around in other directions randomly, these all average out to just a tiny \( V_d \) in one direction.

All the components of this motion add to zero force, except the \( v_d \) component.

\[ \Rightarrow \vec{F} = \int dq \vec{v}_d \times \vec{B} \]

But, all charges have the same net \( v_d \). Only thing changing in integral is \( \int dq = \text{total Q} \). Thus total charge moving in the wire.

That total charge is \( Q = n q v_d A \Delta t \) from above.
\[ Q = I \Delta t = I \frac{\text{length of wire}}{V_d} \]

So,

\[ F = I \frac{\text{length of wire}}{V_d} \hat{V}_d \times \hat{B} \]

The \( \hat{V}_d \)'s cancel, at least in magnitude, and we are left with just the direction of \( \hat{V}_d \). That is along the wires length. So,

\[ F = I \hat{L} \times \hat{B} \]

To come up with this, we had to assume that the wire was straight. For a real, non-straight, wire, we just add up the force from small sections, \( d\vec{F} \), which are straight.

\[ d\vec{F} = I d\vec{L} \times \hat{B} \]

As we've seen over \( \frac{\Delta \vec{F}}{\Delta L} \), the total force is found by integrating all these small elements.

\[ \text{DEMO 68.27 Levitate an aluminum conductor.} \]

Comment on \( +q \) vs. \( -q \) charge carriers in current. Same force either way.
Suppose we had a loop of wire. Here I make it a rectangular loop instead of a circle to simplify the math.

This is immersed in uniform magnetic field pointed into the page (symbolized by $\hat{\theta}$).

What is the force on the loop?

It is the sum of the pieces. Since it is rectangular, we can sum the four sides instead of using an integral.

$$\vec{F}_1 = I h B (\hat{\mathbf{x}}) \quad \text{i.e. to the left as drawn.}$$

$$\vec{F}_2 = I w B (\hat{\mathbf{y}}) \quad \text{i.e. up the page}$$

$$\vec{F}_3 = I h B (\hat{\mathbf{x}}) \quad \text{to the right.}$$

$$\vec{F}_4 = I w B (\hat{\mathbf{y}}) \quad \text{down.}$$

If parallel sides are same length, i.e., a rectangle, then the total force is zero, opposite sides cancel.
We can make this more interesting by adding a small twist. If instead of this we had this:

A clearer, less 3D view, i.e. only side view is:

Then, what is the force?

\[ B \rightarrow I \rightarrow \theta \Rightarrow F_1 = I h B \sin \theta \] into the page.

\[ B \rightarrow I \rightarrow \frac{\pi - \theta}{\pi - \theta} \Rightarrow F_3 = I h B \sin(\pi - \theta) \] out of page

Then pulling loop out as before, just a bit weaker.

Top wire: \[ \rightarrow \bigcirc \Rightarrow F_2 = I w B \] up.

Bottom wire: \[ \rightarrow \bigcirc \Rightarrow F_4 = I w B \] down.

These forces still cancel.
But, there is a net torque!

\[ \tau = \frac{1}{2} I w B \cos \theta + \frac{1}{2} I w B \cos \theta \]

\[ = h w B \cos \theta \]

\[ = h w I B \sin \varphi \]

\[ = I A B \sin \varphi \]

\[ = I \vec{A} \times \vec{B} \]

The torque points out of the page, so the loop rotates counter clockwise.

**DEMO 68.48**
Torque on a current carrying loop.
This torque is the principle behind an electric motor.

- Brushes are aligned with commutator segments
- Current flows into red-colored side of rotor and out of blue-colored side
- Magnetic torque causes rotor to spin counterclockwise

(a) Rotation axis

(b) Rotor has turned 90°
- Each brush is in contact with both commutator segments
- Current bypasses rotor altogether
- No magnetic torque on rotor

(c) Brushes are aligned with commutator segments
- Current flows into blue-colored side of rotor and out of red-colored side
- Magnetic torque again causes rotor to spin counterclockwise

The torque becomes zero when $\dot{A}$ and $\dot{B}$ are parallel, and then points opposite as the loop rotates further. To keep $\dot{E}$ from switching sign, just reverse the current.

A DC motor does that with small "brushes" that alternately contact each end of the loop.

To increase $\dot{E}$, increase $I$ or $A$ (or $B$ of course). Combining multiple loops of wire increases $A$. 
\[ d \vec{F} = I \, d\vec{L} \times \vec{B} \] is also the principle behind a speaker.

A current in the coil causes a force on the coil
\[ \vec{F} = I \, \vec{L} \times \vec{B} \]
because the magnets are arranged to have \( \vec{B} \) perpendicular to the coil everywhere.

The force on the coil pushes the speaker's cone in and out. If \( I \) varies then the force varies and the air pressure from the moving cone varies.

\[ \Rightarrow \text{Varying air pressure} \Rightarrow \text{sound.} \]