Current

So far, we have only considered electrostatics, that is only cases with fixed—unmoving—charges. Now, we'll explore how to describe moving charge \(\rightarrow\) current.

For example, I said that the electric field in a conductor is zero, because if it was not zero the charges will move to make it zero.

We saw that when discharging a capacitor.

![Diagram of a capacitor discharging](image)

The spark was the charges moving to make \(E=0\) in the metal of the screw driver’s blade.

We want to describe this current. Real metals aren’t perfect conductors so the charges don’t move all at once. We want to describe that too.
To hopefully make it more intuitive—conceptually clearer—I'll try to use the analogy of water flow.

Suppose we have a pipe filled with water, and the water is flowing through it.

What causes it to flow is the pressure. For example, a large tank—like a capacitor—could be the source of this pressure.

To make the analogy clearer, let's imagine it as all vertical.

← wide but shallow
← long pipe
The water in the tank is at a higher potential. \[ V = \frac{V}{g} \approx \frac{mgh}{m} = gh \]

Let's define the bottom as zero potential. Then it is like

and the electric potential pushes the charge through the conductor like gravity pushes the water through.

so: 1) Potential (or \( \mathbf{E} \)) is what causes the current.

Next, notice that the water that goes in comes out. It isn't created or destroyed, just pushed around.

so: 2) Charge is conserved in electric current, it just moves around.

How fast does the current flow? Well first, let's look at the water case to note a few things.
The typical speed of the water flowing out of, say, your shower is perhaps 0.1 m/s or so. But recall that our discussion of thermodynamics revealed that matter is made of a large number of molecules that are all moving randomly about, banging into each other. The instantaneous speed of a typical molecule is very high

\[ \frac{1}{2} m \langle V^2 \rangle_{\text{avg}} = \frac{3}{2} kT \]

⇒ a few hundred m/s average speed with collisions every nanosecond or so. (These are for a gas, not a liquid, but the same idea holds).

The actual flow down the pipe is a tiny fraction of this speed. The collisions send each molecule bouncing randomly in different directions, but there is always an additional push from the potential.
It is the same with electric current, the free electrons in a conductor bounce around between the fixed positive charges with a very high speed, $\approx 10^6 \text{ m/s}$. The $E$ field, or potential, adds a tiny non-random motion along the direction of $-E$.

That motion is best called a drift and it is very small, $\approx \text{mm/s}$!

Although the actual charges move slowly, the net effect of the flow is fast. (Recall the rapid spark).

This can be easily seen with the water analogy. Suppose you are taking a shower and someone turns off the water upstream.

You will immediately feel your water flow drop.
If they then turn it back on, you get flow again immediately. Even though an individual water molecule takes very long to move, the pressure travels at the speed of sound $\approx 1$ km/s in water.

Similarly, changes in $\dot{E}$ propagate at the speed of light, but actual charges move very slowly.

so: 3). A conductor, full of charges, has current that is slow drift.

Next, suppose that our pipe was very narrow, how much water flows? Very little. This is how you adjust the flow, by restricting the size of the opening in the pipe.

A big pipe conducts more.

It is easier for us -- or at least more common -- to talk about resistance.

\[ \frac{1}{\text{conductance}} = \text{resistance}. \]
An electrical conductor also allows more current if it is larger cross-section, and we can talk about a conductor's resistance.

So: 4). Real conductors have some resistance to current.

Finally, before doing the math, let's think about current vs. potential.

Notice that the total amount of water flows by each point in the pipe, per second. Mass in = Mass out.

The water doesn't diminish along the pipe, nor does the flow!

What does diminish is the pressure.

The tiny spout at the end would correspond to the same tiny flow everywhere.

So: 5). Potential decreases along a resistive conductor, but current is the same everywhere along it.
OK, now to formally define things.
\[ \frac{dx}{dt} = v \]

\[ I = \frac{dQ}{dt} \]

is the charge moving past a point (or through an area) per unit time.

\[ dQ = q n A v dt \]

\[ \rightarrow \text{dx = distance moved} \]
\[ \downarrow \text{Area} \]
\[ \downarrow \# \text{of charges per area.} \]
\[ \downarrow \text{charge on each moving particle.} \]

So, \[ I = \frac{dQ}{dt} = q n A v \]

Recall that \( v \) is a small drift speed, not the higher instantaneous speed. So,

\[ I = q n A v^d \] to indicate drift speed.
A few comments.

1. $V_d$ is in the direction of the electric field, and should really be a vector. I am not using vector notation here because we don't want to have to mess with it... it is easier to just say "along the wire (i.e. pipe) from positive $V$ to negative $V."$

2. Actually the current is defined to flow from positive $V$ toward negative. That means it is the positive charges that we imagine moving.

As I just said, it is the electrons that actually move in a metal, but we can define it either way. $|E| > 0$ is the direction a + charge would accelerate, so $I > 0$ in the direction positive charge moves. $= -$direction of negative charge.

The direction of the current is actually equivalent, so $I = |q| n A \ v_d \ E$
lb). It is useful to define another quantity

\[ J = \frac{I}{A} = \text{current density} \]

\[ = |q| n \cdot \overrightarrow{V_d} \]

This is now something we can properly treat as a vector:

\[ \overrightarrow{J} = q n \overrightarrow{V_d} \text{ where } \overrightarrow{V_d} \parallel \overrightarrow{E} \]

if \( q > 0 \) and antiparallel if \( q < 0 \).

Either way, \( \overrightarrow{J} \) is in the same direction as \( \overrightarrow{E} \). \( \overrightarrow{J} \) is easier to connect to \( \overrightarrow{E} \), and so the physics. \( I \) is easier to measure, so it is more practical.

Units: \( I = \text{Amps} = C/s \). \( J = A/m^2 \)

Now, we can define resistance.

\[ R = \frac{V}{I} \]

This actually comes from the resistivity, so let's start there.
\[ \rho = \frac{|E|}{|J|} \] is resistivity.

This is Ohm's law \((R = \frac{V}{I} \text{ is a similar statement})\), saying that \(|J|\) is proportional to \(|E|\). Or \(I \propto V\).

Like Hooke's law, and \(|\vec{F}_f| = \mu |\vec{N}|\), this is only empirically approximate, and it is not true for all materials -- like semiconductors.

Units:

\[ \rho = \frac{E}{J} = \frac{V/m}{A/m^2} = \frac{V}{A} \text{ m} = \Omega \text{ m} \]

\[ R = \frac{V}{I} = \frac{V}{A} = \Omega \text{ ohms} \]

Note that \(R = \frac{V}{I}\) and \(\rho = \frac{E}{J}\), Ohm's law is just a statement that \(R\) is not a function of \(V\) or \(I\), sort of like \(F = kx\), Hooke's law is significant in the realization that \(k \propto x\), or friction = \(umN\) and \(m \perp N\).
Some typical numbers.

A flashlight using a 3V battery (2 x 1.5V) draws about 1A of current.

\[ R = \frac{V}{I} = 3 \Omega \]

If this current flows through 1mm diameter wire, how fast do the electrons drift?

\[ I = n q b A v_d \]

\[ v_d = \frac{I}{n q b A} = \frac{1A}{n (1.6 \times 10^{-19})(\pi \times 10^{-6})} \]

\[ = \frac{6 \times 10^{25}}{n} \]

What is \( n \)?

\[ n = \frac{\text{electrons}}{\text{volume}} \approx \frac{\text{atoms}}{\text{volume}} \]

\[ N = \frac{\text{atoms}}{\text{Vol}} = \frac{\text{atoms}}{\text{mol}} \cdot \frac{\text{mol}}{\text{g}} \cdot \frac{\text{g}}{\text{Vol}} \]

\[ = 6 \times 10^{23} \left( \frac{1}{60 \text{ g/mol}} \right) \left( 10 \text{ g/cm}^3 \right) \left( \frac{100^3 \text{ cm}^3}{\text{m}^3} \right) \]

\[ = 10^{29} \text{ electrons/m}^3 \]

\[ \Rightarrow v_d = 6 \times 10^{-4} \text{ m/s} \text{ very slow!} \]
Sample problem:

Imagine three cylindrical copper wires with the same voltage across them. Which has the most current through it? Which the least?

\[
I = \frac{V}{R} \quad \text{or} \quad J = \frac{E}{F} = \frac{V}{\text{Length}}
\]

\[
I = J \cdot \text{Area} = \frac{\text{Area} \cdot \text{Voltage}}{F \cdot \text{Length}}
\]

\[
I_1 = \frac{AV}{FL}
\]

\[
I_2 = \frac{A}{2} \sqrt{\frac{V}{3FL^{1/2}}} = \frac{AV}{3FL}
\]

\[
I_3 = \frac{A}{2} \sqrt{\frac{V}{FL^{1/2}}} = \frac{AV}{FL}
\]

This should just scale by \(3 - \frac{3L}{2} \cdot \frac{1}{2}\)

Good
Example:
Suppose we stretch a wire so that its length increases by a factor of 4, but volume doesn't change. What happens to $R$?

$$R = \frac{V}{I} = \frac{E \cdot L}{J \cdot A} = \frac{E \cdot L}{\frac{E}{s} \cdot A} = \frac{L}{A}$$

$L = 4L$, $V = V = LA$ $\Rightarrow$ $A_f = \frac{1}{4}A$

$$R_f = \frac{\int \frac{4Li}{A_i/4}}{16 \int \frac{Li}{A_i}} = 16 \int \frac{Li}{A_i} = 16 \times R_i$$

Example: Suppose we connect two wires of different diameter.

What is the ratio of the current through them? Equal!
What is the ratio of resistance? Doubles.
What is the voltage along it?
What is the change in electron drift speed? $v$ is twice as high in the narrow part.
Hindenburg: C, V, I, R conspire.

How much charge is required to get a large voltage and so a spark?

\[ Q = CV \]

\[ \rightarrow \text{a few 1000 volts.} \Rightarrow E = 1 \text{ kV/mm is big.} \]

\[ C \text{ for a sphere is } \frac{4\pi \varepsilon_0 R}{\varepsilon_0} \approx 10 \times (10 \times 10^{-12})(100) \approx 10^{-8} \text{ F} \]

\[ \Rightarrow C = 10^{-5} \text{ Coulombs, not enormous.} \]

If \( I = 1 \text{ mA}, \) \( \Rightarrow \) 0.01 seconds to charge that C to 1kV.
Resistance demo.

So, work is done by $E$ moving $q$. Where does it go?

DEMO ball down ramp.

DEMO heat coil.
Temperature dependence of $R$

Empirically, $P$ is a function of temperature. So, of course, must be $R$.

For small variations in temperature,

$$R(T) = R_0 \left[ 1 + \alpha (T - T_0) \right]$$

for conductors, where $\alpha$ is empirically measured for each material.

$$\alpha \approx \text{a few } \times 10^{-3} \ \text{°C}^{-1}.$$  

Tiny, unless $\Delta T$ is big.

This is $\approx$ intuitive in that higher $T$ implies more vibration so more likely to get a hard collision turning it around.

Semiconductors are different, higher temperature can increase the concentration, $n_j$, of charge carriers so $R$ decreases -- dramatically.

$\Rightarrow$ Thermistor.
Superconductivity.

At very low temperatures, some materials have R drop to zero. This is understood to be due to a pairing that makes it quantum-mechanically impossible to lose energy by scattering.

⇒ an ideal conductor.

DEMO.