Review:

Capacitance is another concept related to electric force that is both conceptually useful and practically applicable.

To put it in perspective, we had the Coulomb force, \( F = kq_1q_2/r^2 \), that we could (in principle) use with \( F = ma \) to find motion, but solving real world problems that way would be hopelessly hard so we have developed new related concepts to

a). Find easier mathematical routes to solving problems.

b). Intuitively better understand the phenomena.

These new concepts are:

**Electric Field**: \( \vec{E} = \vec{F}/q_0 \)

where we imagine the force that would be "felt" by a test charge \( q_0 \) but divide out the \( q_0 \) to get an \( \vec{E} \) that can be thought of as existing even if there is no charge there to "feel" the corresponding force.
The electric field is useful because we can think about it for a system (such as a charged plate) in general terms before any charges are put nearby, and field lines give a useful visualization of the system. Particularly easy.

Also, Gauss' law provides a means of finding $\vec{E}$ in symmetrical situations.

**Electric Potential (i.e. voltage)**

A complication with using the electric field is that it is a vector. So we have to determine the various components, which can get messy. We can get around that complication by using energy.

Potential Energy, $U = \text{work that could be done}$, is a scalar and so much easier to calculate. We can also use this with conservation of energy to solve problems.

Electric potential is $V = U/q_0$, again with $q_0$ being some test charge.
An additional simplification with using electric potential is that we can define the zero point any way we want since only differences in energy matter.

If we need to get $\vec{E}$ just use

$$\vec{E} = -\nabla V$$

and $\vec{E} = E_0$.

**Capacitance:** So $V$ is related to energy, well energy over charge. Now, we developed a new concept, that further relates to energy.

$$C = \frac{q}{V}$$

measures how the potential increases as we add charge, and as we'll see, it is related to how much energy is stored in the collected charge.
So, \[ C = \frac{q}{V} \]

and we saw last time how to calculate \( C \) by finding \( V \) for some configuration.

A pair of plates is a very good example.

\[ \begin{array}{c}
\quad A \\
\uparrow \\
\quad +q \\
\end{array} \quad E = \frac{\sigma}{\varepsilon_0} \]

\[ \begin{array}{c}
\quad A \\
\uparrow \\
\quad -q \\
\end{array} \quad \Rightarrow V \text{ between plates is} \]

\[ V = \int_0^d \frac{\sigma}{\varepsilon_0} \, dy = \frac{\sigma \cdot d}{\varepsilon_0} = Ed \]

\[ = \frac{q}{A} \cdot \frac{d}{\varepsilon_0} \]

\[ \therefore C = \frac{q}{V} = \frac{\varepsilon_0 A}{d} \]

How much energy is stored on such a capacitor? Well it depends on how much charge is "on it."

Capacitance is not energy -- it is related to capacity to store energy. Only by adding charge is that capacity put to use.
It might be useful to consider a gravitational analog: A dam.

The capacitance of the dam is related to $A$.

The bigger $A$ is, the more charge (mass of water) we can add without greatly increasing the height of the water.

$U = mgh \Rightarrow V = \frac{U}{m} = gh$  c.f. $V = Ed$

If $A$ is small, adding water makes a larger $h$ ($\Rightarrow V$) than if $A$ is large.

$\Rightarrow$ I can intuitively think of the "volume" of the dam as it's capacitance.

Large $C$ $\Rightarrow$ adding mass doesn't increase the depth much

adding charge doesn't increase voltage much.

A dam stores energy (mass held at high potential).

A capacitor stores energy (charge held at high potential).
We can't stretch this analogy too far, but it is somewhat useful.

For example, two capacitors in parallel have the same potential across them would correspond to two tanks (i.e. dams) that are connected.

In series is hard (impossible?) to see with "tanks" since it relies on polarization to make capacitors in parallel have the same charge.

But, we can use the analogy to see how energy is stored in a capacitor. The more water I add to the dam, the more energy I get stored in it. And the potential grows making subsequently added water store even more energy per kg -- because I have to lift it higher to stack it on top of the water already there.
So, similarly adding a little bit more charge to a capacitor requires "pushing" it over the potential \( V \) already on the capacitor.

\[
dU = V \, dq = \frac{q}{C} \, dq
\]

So,

\[
U = \int dU = \int_{Q_0}^{Q} \frac{q}{C} \, dq = \frac{Q^2}{2C}
\]

But the voltage on \( C \) with \( Q \) is given by

\[
Q = CV
\]

or \( U = \frac{1}{2} QV \)

So,

\[
U = \frac{C^2 V^2}{2C} = \frac{1}{2} CV^2
\]

Note: \( Q = \) total charge & we used \( q \) to integrate up to it. Similarly \( V \) here is the final \( V \).

This \( U \) is the total energy stored in a capacitance of \( C \) having total charge \( Q \) or voltage \( V \).
Let's calculate the energy in our demo capacitor from yesterday.

\[ U = \frac{1}{2} \times (10^{-3} \text{F}) \times (400^2) \]

\[ = \frac{16}{2} \times 10^{-3} \times 10^4 \]

\[ = 80 \text{ J} \]

which is enough to lift me \( \approx 1 \) foot off the ground. \((80 \text{ kg} \times 10 \frac{m}{s^2} \times 0.1 \text{ m})\).

This is similar to some homework problems.

Let's try one other example, a defibrillator. They are basically like our demo cap, designed to pump energy into a capacitor in some slow, manageable way and then rapidly dump that energy in a fast jolt. You want about 10 times the energy of above and about 10 times the voltage. So what capacitance?

\[ U = 800 \text{ J} = \frac{1}{2} \times C \times (4000)^2 \]

\[ \Rightarrow C = \frac{1600}{16 \times 10^6} = 10^{-4} \text{ F} \]

\( \frac{1}{10} \)th the \( C \) of above.

Aside: Power = Energy/time \( \approx \frac{800 \text{ J}}{1 \text{ ms}} = 800 \text{ kW} \)!
It is worth thinking about where the energy is stored. We can say that it is stored in the electric field. This may seem just philosophical, but it comes in handy later.

To see this, let's calculate the energy density: energy per volume, for parallel plates.

\[ u = \frac{U}{A d} = \frac{C V^2}{2 A d} \]
\[ = \left( \frac{\varepsilon_0 A}{d} \right) \frac{V^2}{2 A d} \]
\[ = \frac{1}{2} \varepsilon_0 \left( \frac{V}{d} \right)^2 \]
\[ = \frac{1}{2} \varepsilon_0 E^2 \]

This is actually true generally, not just for the parallel plate case.

This means we can think of the presence of an electric field at some point in space as corresponding to some energy density -- electric potential energy per unit volume.
Dielectrics

Recall that we thought last time about what happens if we insert a conductor into a parallel plate capacitor.

\[ \begin{array}{c}
+2 \\
-2
\end{array} \quad \begin{array}{c}
+8 \\
-8
\end{array} \Rightarrow \text{charges induced on opposite sides}

This makes it into two capacitors in series, so the total C decreases.

Now, what happens when we insert an insulator?

The charges can't move to the surfaces of the insulator, but they can polarize. That is, if there is any asymmetry in how the + and - charge is held in the molecules, they can align themselves.
If the molecule is like an electric dipole,

then there will be a net torque that will rotate it into a polarized orientation, this slightly decreases the electric field, so the capacitance increases.

Some details: Real molecules are not just $\pm$ -dipoles like I have drawn, but they have some effective dipole moment because the electrons are more strongly pulled, chemically, toward one side.
Why does it increase rather than decrease the capacitance?

Two views: math & conceptual.

Mathematically: \( C = \varepsilon_0 \frac{A}{d} \)

That came from,

\[ C = \frac{q}{V} = \frac{q}{Ed} \]

The induced polarization reduces \( E \), so it increases \( C \).

Conceptually: \( C \) measures the potential I get as I add charge. A larger capacitance means less voltage for the same \( q \), because some of the \( V = \frac{U}{q} \) goes into rotating the dipoles into a polarized state.

\[ \text{Less } V \text{ needed per } q \Rightarrow \text{ bigger } C. \]
We don't want to have to model the dipole moments to figure out how C changes, instead, we can just measure it and tabulate constants for different materials.

\[ K \equiv \text{dielectric constant} \]

\[ C = KE_0 \frac{A}{d} \]

where \( K \) has no units and is just the ratio of \( C \) with the material to \( C \) with vacuum.

Finally, we can see why we have been carrying around the sub-\( 0 \) in \( \varepsilon_0 \).

\[ \varepsilon \equiv KE_0 \]

So,

\[ C = \varepsilon \frac{A}{d} \]

Real Capacitors take advantage of this to increase \( C \) by factors of 10 or so; or make a given capacitance in a smaller package.
One problem though is that if $E_1$ gets too big the material suffers breakdown. Recall that while discussing lightning we said that air breaks down (ionizes) at about 30 MV/m. Other dielectrics also have some limit.

Squeezing $d$ to increase $C$ means $E$ must be higher for a given $V$.

$$E = \frac{V}{d}$$

This puts a limit on how much voltage you can put across the cap before it pops.
Sample problem:

\[
\begin{array}{c}
\frac{A}{k_1} ; \frac{A}{k_2} = d \\
\end{array}
\]

What is \( C \)?

\[
C = k_1 \varepsilon_0 \frac{A}{d} + k_2 \varepsilon_0 \frac{A}{d} \text{ in parallel.}
\]

\[
= \varepsilon_0 \frac{A}{d} \left( \frac{k_1 + k_2}{2} \right)
\]

How about this \( \frac{A}{k_1} ; \frac{A}{k_2} = d \)?

\[
\frac{1}{C} = \frac{1}{k_1 \varepsilon_0 A / d} + \frac{1}{k_2 \varepsilon_0 A / d}
\]

\[
= \frac{d}{2 \varepsilon_0 A} \left( \frac{1}{k_1} + \frac{1}{k_2} \right)
\]

\[
= \frac{d}{2 \varepsilon_0 A} \left( \frac{k_2 + k_1}{k_1 k_2} \right)
\]

\[
\Rightarrow C = \frac{\varepsilon_0 A}{d} \cdot 2 \left( \frac{k_1 k_2}{k_1 + k_2} \right)
\]