Electric Fields

Last time we talked about electric charge and electric force. A brief review is:

\[ \vec{F} = k \frac{q_1 q_2}{r^2} \]

$q$ in Coulombs, $k$ is very large.

We talked about motion of charge and saw:

- Conductors
- Insulators
- Induced charge
  - Polarization
- Charge distributes on surface of conductor.

A demo: ping-pong ball between paddles. What will happen?

1. Weak attraction due to induced charge
2. Charge transfer
3. Repulsion
4. Repeat
We also did one example of calculating the force from an extended, charged object.

Coulomb's law only applies to point objects.

\[ \vec{F}_1 \rightarrow \vec{F}_2 \leftarrow \]

\[ q_1, \quad q_2 \]

That is way too simple to be practically useful, real life objects have size.

We did:

A point charge and a line of charge.

Review idea.

So, how would we calculate the force on the ping-pong ball? We'd have to add the force from all the charges on each plate acting on each charge on the ball.

That is a mathematical nightmare. We need some tricks...
First trick is what we'll discuss today, other will follow.

So, even without the whole messy calculation, we can predict what will happen: a bunch of charges will push on the ball one way, then the other. That is much like using \( \frac{mg}{r^2} \) to describe \( \vec{F} = -\frac{GM_{\text{Earth}} m}{r^2} \).

We didn't have to use that mess every time we dropped a ball, we just knew that things get pulled down everywhere.

That \( \vec{g} \) is an example of a field. (Star Trek---and other such---often refers to "force fields." That is a bad name, but perhaps a useful analogy).

\( \vec{g} \) is an acceleration field. It is related to the gravitational force as

\[
\vec{g} = \frac{\vec{F}}{m}
\]
In the same way, the electric field is 

\[ \vec{E} = \frac{\vec{F}}{q} \]

That is, \( \vec{E} \) is related to the force that would be exerted on a charge \( q \) if it were there. 

\( \vec{F} \) is related to the force that would be exerted on a mass \( m \) if it were there.

OK, enough analogic introduction... my point was just to set the stage that this is not particularly new. Now let’s look at the details.

Two point charges

\[ \vec{F}(x,y) \text{ on } q \text{ is} \]

\[ \vec{F} = k \frac{Qq}{r^2} \]

The electric field is

\[ \vec{E}(r) = \frac{\vec{F}(r)}{q} = k \frac{Q}{r^2} \]
That doesn't look particularly useful, but it is a conceptually important difference. I've removed $q$, so I only have to think about one half of it. (Trivial for this case, but not others...).

Q produces an electric field.

$q$ responds to that field.

This separation is a useful conceptual simplification (as well as mathematically).

Suppose we don't know $Q$ (or how it is distributed), we could measure $\mathbf{E}(\mathbf{r})$, that is measure its magnitude and direction at each point $\mathbf{r}$, and make a map.

Arrows show $\mathbf{E}$ at various $\mathbf{r}$.

That map shows the force that a unit charge would experience at each point.
It is called a field because it depends on position.

It is a "vector field" because it is a vector quantity.

\( \vec{g} \) is also a vector field. We didn't have to think much about that because it doesn't change much with position (a pretty flat field) and it points in a pretty consistent direction, Down.

So, all I need is to know \( m \) and \( \vec{g} \). I can easily get the magnitude and direction of the gravitational force.

Similarly, once I know \( q \), I can easily get the magnitude and direction of the electrical force.

\[ \vec{F} = q \vec{E} \]
In the case of the ping-pong ball, $\vec{E}$ is approximately uniform as a function of position and all pointed right. (Looks a lot like $\vec{g}$, eh?)

A couple details:

- We have to assume that the object "feeling" the electric field is small enough that $\vec{E}$ doesn't change across it. (We assumed this in $\vec{mg}$ too).

- We have to assume that the charge is so small that it doesn't change the electric field. Such as
The same is true for gravity, it is just so weak that it is obviously ignorable.

What isn't true in gravity is polarization. A large charge can induce polarization around it. So, it must be small...

- We also have to worry about signs. So, $\vec{E}$ is calculated assuming a positive test charge.

$$\vec{E} \rightarrow -\vec{E}$$

This doesn't mean the field is only valid for positive charges, just that it is defined that way. When using $\vec{E}$ to find the force on a actual charge, the signs work.

$$\vec{F} = (-q)\vec{E}$$

$$\vec{F} \rightarrow -\vec{E}$$

$$+Q \rightarrow -Q$$
So, let's calculate some electric fields.

First, a single point charge - easy.

\[ \vec{E}(\vec{r}) = k \frac{Q}{r^2} \hat{r} \]

\[ = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} \hat{r} \]

This is as good of a point as any to switch from \( k \) to \( \frac{1}{4\pi \varepsilon_0} \). We just define

\[ k = \frac{1}{4\pi \varepsilon_0} \quad \text{where} \quad \varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2} \]

\( k \) makes sense as a simple proportionality constant, and \( \frac{1}{4\pi \varepsilon_0} \) is messier. We'll see later that this is actually more convenient because the \( 4\pi \) cancels (think sphere's surface area) and \( \varepsilon_0 \) becomes \( \varepsilon \) for media other than vacuum.

So,

\[ \vec{E}(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} \hat{r} \]
Now, what is the field for two charges?

\[ E = E_+ + E_- = \left[ \frac{1}{4\pi \varepsilon_0} \frac{Q}{(x - \frac{d}{2})^2} - \frac{1}{4\pi \varepsilon_0} \frac{Q}{(x + \frac{d}{2})^2} \right] \]

\[ E_x = \frac{Q}{4\pi \varepsilon_0 x^2} \left[ (1 - \left( \frac{d}{2x} \right)^2) - (1 + \left( \frac{d}{2x} \right)^2) \right] \]

Messy. But we can use a simplifying approximation if \( \frac{d}{x} \ll 1 \). I.e. \( x \gg d \).

Then, by the binomial theorem:

\[ (1 + y)^n = 1 + n y + \frac{n(n-1)}{2!} y^2 + \ldots \]

\[ \approx 1 + ny \quad \text{for} \quad y \ll 1 \]

\[ \Rightarrow E_x \approx \frac{Q}{4\pi \varepsilon_0 x^2} \left[ 1 + \frac{d}{x} - (1 - \frac{d}{x}) \right] \]

\[ = \frac{Q}{4\pi \varepsilon_0 x^2} \frac{2d}{x} = \frac{Qd}{2\pi \varepsilon_0 x^3} \]

\( Qd \) is called the electric dipole moment.

\[ \mathbf{p} = Q \mathbf{d} \]

direction is neg to pos \( Q \).
It turns out that this $\frac{1}{r^3}$ behavior holds even off the x-axis.

A couple things to note:

$\frac{1}{r^3}$ is smaller than $\frac{1}{r^2}$ -- good!

Can measure $p = Qd$ e.g. of some molecule.

OK. Moving on, how about four charges.

This is called an electric quadrupole.

$$E = \frac{3Qd^2}{2\pi \varepsilon_0 x^4}$$  \hspace{1cm} \text{I won't derive, but same.}

(need to keep 3 terms in expansion).

OK. Moving on... how about an infillite charge distribution...
Electric Field from infinite line charge

\[ \frac{dQ}{ds} \]

\[ dQ = \lambda ds \]

\[ \vec{E} = E_x \hat{x} \]

\[ E_x = \int dE_x = \int \frac{1}{4\pi \varepsilon_0} \frac{dQ}{(x+s)^2} \]

\[ = \frac{1}{4\pi \varepsilon_0} \int_0^\infty \frac{\lambda ds}{(x+s)^2} \]

\[ = \frac{\lambda}{4\pi \varepsilon_0} \left[ \left. -\frac{1}{x+s} \right|_0^\infty \right] = \frac{\lambda}{4\pi \varepsilon_0} \left[ 0 - \left( -\frac{1}{x} \right) \right] \]

\[ = \frac{\lambda}{4\pi \varepsilon_0} \frac{1}{x} \]

How about this configuration?

\[ dE_x = \frac{1}{4\pi \varepsilon_0} \frac{dQ}{r^2} \cos \theta \]

\[ E_x = \frac{1}{4\pi \varepsilon_0} \int \frac{2dy}{(x^2+y^2)^{3/2}} \]

\[ E_x = \frac{\lambda}{4\pi \varepsilon_0} \left[ \frac{y}{x^2+x^2+y^2} \right]_{-\infty}^{\infty} = \frac{\lambda}{4\pi \varepsilon_0} \frac{1}{\sqrt{(x^2+y^2)^2+1}}_{-\infty}^{\infty} \]

\[ = \frac{\lambda}{4\pi \varepsilon_0} \left[ 1 - (-1) \right] = \frac{\lambda}{2\pi \varepsilon_0} \]
Electric Field Lines

So, we have a (barely) simplified calculation with \( \vec{E} \) instead of \( \vec{F} \). But it is conceptually appealing to think about \( \vec{E} \) field instead of forces. "Electric field lines" are a useful way to visualize these.

Our infinite line of charge:

The field lines for a point charge:

The field lines for a dipole:

\( + \) to \( - \) or to infinity.

How would it change with a \( +\) and \( -\)? (cf ping pong ball).
DEMO field lines.

→ Not necessarily path of a charge -- only one starting from rest. (Compare to gravity).

These are visualized using a powder suspended in oil. We can think of this powder as a "bunch" of dipoles. What happens to a dipole in an electric field?

\[ \vec{F} = q \vec{E}, \quad \text{and} \quad \vec{F} = -q \vec{E} \]

⇒ A torque

\[ |\vec{r}| = \frac{d}{2} q E \sin \Theta + \frac{d}{2} q E \sin \Theta = d q E \sin \Theta \]

= \( \vec{P} \times \vec{E} \) pointed into the page

⇒ dipole moment, \( \vec{d} \) and right-hand rule.

The \( \vec{E} \) field "torques" the dipole into line.
Of course, it will gain angular velocity as it rotates into line -- and will overshoot then be torqued back. Sound familiar? Oscillation. I could try to calculate the frequency of osc., but for later, it is more useful to look at the energy it has, i.e., potential energy as it starts rotating.

\[ U = -W = - \int_{\text{}}^{\theta} T \, d\theta = +\int_{\pi/2}^{\text{}} p E \sin \theta \]

\[ = -pE \cos \theta \]

\[ = -\hat{p} \cdot \hat{E} \]

(A taste from the past, and a hint for the future -- we'll see B fields in magnetism).
Millikan's Oil Drop experiment.

Robert Millikan measured the charge of an electron using a clever electric setup. Spray tiny oil drops.

By timing motion of drops can determine \( F_E \Rightarrow G \) and see that it comes in steps of size \( e \). (\( 1.6 \times 10^{-19} \text{C}, \text{not} \ 2.818 \)).