Waves part 2

Last time we discussed the mathematical representation of a wave.

A wave is a disturbance that propagates.

More specifically, a transverse wave has the disturbance occurring transverse to (i.e. perpendicular to) the direction that the wave propagates. So, we describe it with $y(x,t)$

$\Rightarrow$ direction of propagation

$\Rightarrow$ transverse direction—of disturbance

We found that any $y(x,t)$ satisfying

$$y(x,t) = f(x-\nu t) \text{ or } f\left(\frac{x}{\nu} - t\right)$$

is a wave propagating with speed $\nu$.

This makes sense since it means that $y$ at one $x$ and $t$ is repeated at a different $x$ some time later.
We looked specifically at sinusoidal, periodic waves. They are mathematically simple cases.

\[ y(x,t) = A \cos(kx - wt) \]

\[ k = \frac{2\pi}{\lambda} \quad \quad \quad \omega = \frac{2\pi}{T} = 2\pi f \]

\[ = \frac{2\pi f}{v} \]

\[ \rightarrow \text{since } v = \frac{\lambda}{T} = \lambda f \]

Can rewrite as

\[ y(x,t) = A \cos \left( \frac{2\pi f}{v} (\frac{x}{v} - t) \right) \]

\[ = A \cos \left( \frac{2\pi f}{v} (x - vt) \right) \]

\[ = A \cos \left( k(x - vt) \right) \]

So, what if \( v \) is less than zero?

\[ y(x,t) = A \cos \left[ k(x + \frac{1}{v} t) \right] \]

So

\[ y(x,t) = A \cos(kx + \omega t) \quad \text{is moving left.} \]
So far, we have only talked about transverse waves. **Longitudinal waves** are where the medium's disturbance is in the same direction as the wave's propagation.

Demo with a spring.

We can similarly describe this mathematically. But now, it is

\[ x(x,t) \quad \text{not} \quad y(x,t) \]

since the particles in the medium oscillate along \( x \). Having \( x \) be a function of itself is a bit confusing, but probably easier if you think of pressure waves.

\[ P(x,t) \] makes air molecules oscillate along the direction the wave is travelling.

We'll deal with these next week as sound waves.
Power transmitted by waves.

Waves transmit energy. It is easiest to see how by viewing an element of a string— as we did to determine the velocity.

The net tension pulls up on the leading edge and down on the peak causing the shape to propagate.

There is energy transmitted because work is done.

Recall \( \text{Work} = \text{Force} \times \text{distance} \)

(Well, \( dW = Fdx \) to be exact).

The net tension is a force that causes the string to move, so it does work; so energy is transferred.
The rate of energy transfer, i.e., the power is
\[ P = \frac{1}{2} \sqrt{\mu F} \omega^2 A^2 \]

average power
\[ \Rightarrow \text{Tension} \]

for a sinusoidal wave on a string with tension F. (The \( \frac{1}{2} \) is because it is average of \( \sin^2 \)).

The text derives that -- in a somewhat arcane way. I think it suffices to see it -- that is make it intuitive -- with the kinetic energy part.

\[ K = \frac{1}{2} m v^2 \Rightarrow dK = \frac{1}{2} dm v^2_y \]
for a small piece of the string.

\[ v_y = \frac{\partial y}{\partial t} = \frac{\partial}{\partial t} (A \cos (kx - \omega t)) = \omega A \sin (kx - \omega t) \]

\[ \Rightarrow dK = \frac{1}{2} dm \omega^2 A^2 \sin^2 (kx - \omega t) = \frac{1}{2} m \omega^2 A^2 \sin^2 (kx - \omega t) \]

\[ \Rightarrow \frac{dK}{dt} = \frac{1}{2} m \frac{\partial}{\partial t} A^2 \omega^2 \sin^2 (kx - \omega t) \]

\[ = \frac{1}{2} m \nu^2 \omega^2 A^2 \sin^2 (kx - \omega t) \]
But, as you may remember from oscillations last quarter, kinetic energy is only part of the energy, there is also potential energy in the stretching of the string. In fact, it is half-and-half. (I won't prove).

So,

\[ \frac{dE}{dt} = 2 \frac{dK}{dt} = \mu \nu \omega^2 A^2 \sin^2 (kx - \omega t) \]

and

\[ P_{\text{avg}} = \frac{1}{2} \mu \nu \omega^2 A^2 \]

Same as textbook's result since \( \nu = \sqrt{\frac{F}{\mu}} \).

Note: the important thing is not just to memorize the formulas, but to understand what they mean and where they come from. So:

Oscillating mass has energy

As this oscillation moves with the wave energy is transferred to the new position. A tsunami is a dramatic example.
So when a wave hits a boundary where does the energy go?

Demo with waves on string and spring.

End point fixed so no matter what F there is no motion so no work. How can energy be transferred?

By Newton's 3rd law, the thing holding it fixes has a reaction force back on the string pushing it down so a reflected (inverted) wave bounces the energy back, with the same amplitude (and hence the same power).

It is just like an echo. (Well actually an echo is a reflected wave -- a longitudinal pressure wave -- so an echo is "just like it.")

Demo with the torsional wave machine.

Free end, Fixed end, damping, spring too
Partial reflection

Light string    Heavy string

Partial reflection occurs, and partial transmission.

moves slower

If wave moves from heavy to light, not inverted.

DEMO with wave machine.

Impedance matching is an example of this. That's why CPU clock can be faster than the bus.
CDF's bus e.g. is "only" 53 MHz.

Note: This reflection issue is NOT relevant to your "three rope" homework problem. (why not?)
Principle of superposition

Two waves add to make a net wave.

\[ y(x, t) = y_1(x, t) + y_2(x, t) \]

for all \( x \) and all \( t \). This is nothing new, really. We know forces add and waves are just caused by forces, so they should add too.

This is easy to demo on our wave examples.

Another example is "adding" sound waves, this is usually referred to as "mixing" but it is just adding (perhaps with different volume control i.e., changing the amplitude).

Can you subtract waves? E.g., can you subtract sound? Yes, just sense, invert \( (A \rightarrow -A) \) and add. Sound travels slow, so that is easy. (Noise cancelling headphones, or noise cancelling microphone).
Once you’ve added two waves, can you re-extract them? [DEMO wave machine -- separated waves]

Well, if I mixed music and voice, you can tell them apart -- or say a soprano and a tenor. But you are intelligent, can a machine separate them?

Yes

\[ y(x,t) = A_1 \cos(k_1x - \omega_1t) + A_2 \cos(k_2x - \omega_2t) \]

\[ = A_1 \cos[2\pi f_1 (\frac{x}{v} - t)] + A_2 \cos[2\pi f_2 (\frac{x}{v} - t)] \]

Different frequencies

You could select just the frequency you want.

AM radio is many stations added, your receiver just filters away everything but the selected frequency.

Example of tsunami wave.
Water Column Height at 51407

Gulf of Alaska Earthquake, Ms=7.6, 6 March 1988, 22:36 GMT

Unfiltered Data (Tide Signal Present)
Seismic surface waves
Tide Signal
Tsunami waves

Filtered Data (Tide Signal Removed)
Seismic surface waves
Tsunami waves

Time after main shock
Water waves are a mix of transverse and longitudinal.

[Diagram of transverse and longitudinal waves]

Water moves up & down as well as forward and backward.

The speed of the wave is \( \propto \sqrt{D} \) depth.

As it moves into shallow water, the front slows and the back catches up, becoming

and the amplitude increases correspondingly, in fact \( A \propto \frac{1}{\sqrt{D}} \). Much bigger!
Wave interference

Suppose I sent two waves down a string where both have the same amplitude and frequency but different phase. They add to

\[ y(x,t) = A \cos(kx-\omega t) + A \cos(kx-\omega t + \phi) \]

\[ = 2A \cos \left( \frac{\phi}{2} \right) \cos \left( kx-\omega t + \frac{\phi}{2} \right) \]

[used trig identity: \( \cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2} \)]

\[ = \left[ 2A \cos \frac{\phi}{2} \right] \cos \left( kx-\omega t + \frac{\phi}{2} \right) \]

\[ \uparrow \text{New amplitude} \quad \uparrow \text{Phase offset.} \]

Two waves -- same \( A \) & \( \omega \) and direction---but different phase \( \Rightarrow \) New wave.

\( \phi = 0 \Rightarrow 2A \quad \Rightarrow \text{constructive interference} \)

\( \phi = \pi \Rightarrow 0 \quad \Rightarrow \text{destructive interference} \) (i.e. like noise cancelling).
The text works out the case of inverted waves moving in the opposite direction.

\[ y(x,t) = A \cos(kx - wt) - A \cos(kx + wt) \]

\[ = 2A \sin(kx) \sin(wt) = 2A \sin(wt) \sin(kx) \]

Since \( \cos(a + b) = \cos a \cos b - \sin a \sin b \)

\[ y(x,t) = A \left\{ \frac{\cos(kx \cos(wt) + \sin(kx \sin(wt))}{\cos(kx \cos(wt) + \sin(kx \sin(wt))} \right\} \]

So, we no longer have a traveling wave. No \( kx - wt \)!
Just separate products. So it is a standing wave.

![Amplitude varies with time](image)

Such a standing wave is rather easy to produce using reflections. Send \( A \cos(kx - wt) \)
down a string to a fixed point, and inverted wave reflects back, \( \Rightarrow -A \cos(kx + wt) \).

But, I need the reflection point exactly one wavelength away...
If I have a reflection point \( L \) away, then make \( z = L \). But I don't directly control \( z \).

Well, I can change \( f \), and \( z = \frac{v}{f} = \sqrt{\frac{T}{m}}/f \)

So at the "magic" \( f \) I will get reflected waves that just match in phase to give

\[-A \cos(kx + wt).\]

**DEMO standing waves**

Actually, I can get a standing wave with even \( \frac{L}{2} \).

\[y\]

\[\text{ Makes sense, right?} \]

The "half" wave is reflected inverted too.

In fact, it works for any \( L \) such that the \( \sin kx \) factor in front = 0. (Meaning we have a node at \( x=L \).

\[\Rightarrow \sin kx = 0 = \sin \frac{2\pi x}{L}\]

\[\Rightarrow x = 0, \frac{L}{2}, \frac{2L}{2}, \frac{3L}{2}, \ldots\]

\[\text{Lift end}\]
So if \( L = \frac{L}{2} \), I get a standing wave like this.

\[ y \uparrow \]

\[ L \]

\[ x \]

If \( L = \lambda \), then

\[ y \uparrow \]

\[ L \]

\[ x \]

Etc.

What are the corresponding frequencies?

\[
L = \frac{L}{2} \quad \Rightarrow \quad L = \frac{\lambda}{2} \quad \Rightarrow \quad L = \frac{V}{2f} = \frac{k}{L} \sqrt{\frac{T}{m}} \quad \Rightarrow \quad f = \frac{k}{2L} \sqrt{\frac{T}{m}}
\]

and

\[
L = \lambda \quad \Rightarrow \quad f = \frac{1}{L} \sqrt{\frac{T}{m}}
\]

Question: Is the string ever flat? 
" " " " " " at rest?
These frequencies are called harmonics. The first is the fundamental frequency, and just like the "resonant frequency" we discussed with simple harmonic motion, it is a "natural" frequency for the object so the small perturbations added by the drive build up.

This can happen in many physical setups. Many are the basis for music, but others:

- Normal modes of the earth cause it to ring after an earthquake.
- The sun.
- Waves in Saturn's rings.
- Tides in certain bays.
- Tidal phase lock of the moon.
- Audio feedback... sort of.