Example: a spring.

\[ F = -kx \]

Always pulls back to equilibrium. For gravity, \( k = \frac{GMm}{r^2} \).

For spring \( k \) is intrinsic to the spring.

\[ F = ma = m \frac{d^2x}{dt^2} = -kx \]

\[ \Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \]

That is a differential equation, which we don't know how to solve, yet. But it is a very common form. ... the result of any linear restoring force (e.g. a hole through the earth).

The solution is

\[ x(t) = A \cos(\omega t + \phi). \quad \text{What is } A \text{ and } \omega? \]

Well, \( \frac{d^2x}{dt^2} = \frac{d}{dt} (\omega A \sin(\omega t + \phi)) = -\omega^2 A \cos(\omega t + \phi) \)

\[ \Rightarrow (-\omega^2 + \frac{k}{m})A\cos(\omega t + \phi) = 0 \]

\[ \Rightarrow \omega = \sqrt{\frac{k}{m}} \]

\[ f = \frac{\omega}{2\pi} \]
What is \( w \)? We used that symbol for angular speed before. That is suggestive...

When \( \omega t \) goes from 0 to \( 2\pi \), \( x \) returns to where it started.

\[
\cos (\phi) = \cos (2\pi + \phi).
\]

So the "period" -- time for one complete cycle -- is \( T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{m}{k}}} \).

\( \phi \) is the "phase angle." When \( t=0 \), \( x(t) = A \cos \phi = x_0 \).

So,

![Diagram of a cosine wave with labeled time axis and initial value]
Of course if I pulled the spring to $x_0$ and set go, it would not follow that path. What would I need? A $v_0$. Drawn

$$v(t) = \frac{dx}{dt} = -Aw \sin (wt + \phi)$$

$$\implies v_0 = -Aw \sin \phi$$

or $$\frac{v_0}{x_0} = -\frac{Aw \sin \phi}{A \cos \phi} = -\frac{\omega \tan \phi}{\sqrt{\frac{m}{k}}}$$

$$\implies \phi = \arctan \left( -\frac{v_0}{\omega x_0} \right) = \arctan \left( -\frac{v_0}{x_0} \sqrt{\frac{m}{k}} \right)$$

So, we have $\phi$ and $\omega$ in terms of $v_0$ and $x_0$. How about $A$?

$$v_0 = -Aw \sin \phi \implies \left( \frac{v_0}{\omega} \right)^2 = A^2 \sin^2 \phi$$

$$\frac{1}{2}$$

$$x_0 = A \cos \phi \implies \frac{x_0^2}{A^2 \cos^2 \phi} = \frac{v_0^2}{\omega^2} + x_0^2$$

$$\implies A^2 \left( \sin^2 \phi + \cos^2 \phi \right) = \frac{v_0^2}{\omega^2} + x_0^2$$

$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$$

$$a(t) = ? = \frac{d}{dt} v(t) = -Aw^2 \cos (wt + \phi) = -\omega^2 x(t)$$
Summary

\[ \frac{d^2x}{dt^2} + \frac{K}{m} x = 0 \quad \Rightarrow \quad \text{SHM} \]

\[ x(t) = A \cos(\omega t + \phi) \]

\[ \omega = \sqrt{\frac{K}{m}} \quad \Rightarrow \quad T = \frac{2\pi}{\omega} \]

\[ T = 2\pi \sqrt{\frac{m}{K}} = \frac{2\pi}{\sqrt{\frac{K}{m}}} \]

\[ x_0 = A \cos \psi \quad \Rightarrow \quad \psi = \arctan \left( -\frac{v_0}{x_0 \sqrt{\frac{m}{K}}} \right) \]

\[ v_0 = -\omega A \sin \psi \quad \Rightarrow \quad A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}} \]

In the example of falling through the earth,

\[ F(r) = -\frac{GMm}{R_E^3} v \quad \Rightarrow \quad K = \frac{GMm}{R_E^3} \]

\[ T = 2\pi \sqrt{\frac{m}{GMm/R_E^3}} = \frac{2\pi R_E^3}{\sqrt{GM}} \]

That is what we found for Kepler's third law in an orbit. It takes as long to fall through as to fall around... weird. But interesting.
A projection onto 1D of uniform circular motion is SHM.

\[ x(t) = A \cos(\omega t + \phi) \]
\[ y(t) = A \sin(\omega t + \phi) \]
\[ x^2 + y^2 = A \]

"Projection" onto one dimension is \[ x(t) = A \cos(\omega t + \phi) \]. \[ \frac{dx}{dt} \] is larger as it sweeps past 0. \[ \frac{dx}{dt} \] is larger at \[ \frac{\pi}{2} \] (like the search light on a light house).

Falling thru the earth has same \[ x(t) \] as falling around.

Eg. \[ \frac{dx(t)}{dt} = 0 \] when \[ t = 0 \] in both cases

\[ \frac{dx(t)}{dt} = \text{max} = \text{orbital speed at center} \]
What about vertical motion of a spring?

\[ F_{\text{net}} = -ky - mg \]

With gravity, new equilibrium point is \( d \) where

\[-kd - mg = 0 \quad \text{and} \quad kd = mg\]

So, now if I move the mass down another distance \( x \),

\[ F = -k(-d + x) - mg \]

\[ = kd - kx - mg \]

\[ = mg - kx - mg \]

\[ = -kx \]

Behaves as if there is a new unstretched equilibrium point.
Pendulum

\[ F = -mg \sin \theta \]

\[ = -mg \theta \text{ if } \theta \text{ is small.} \]

\[ = -mg \frac{x}{L} \]

\[ = -\frac{mg}{L} x \]

This is just like a spring with \( k = \frac{mg}{L} \).

\[ \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg}{mL}} = \sqrt{\frac{g}{L}} \]

\[ f = \frac{1}{2\pi} \sqrt{\frac{g}{L}} \]

\[ T = \frac{2}{f} = 2\pi \sqrt{\frac{L}{g}} \]

Period does not depend on the mass, but it does depend on \( g \).

Can measure \( g \).

Can use for a clock since \( T \) of amplitude.
For that simple pendulum, we ignored the mass of the string.

A more general case is called a "physical" pendulum as opposed to "simple."

\[ I = -mgd \sin \theta \quad \text{Restoring torque} \]

\[ \tau = -mgd \theta \]

\[ I = I \alpha = I \frac{d^2 \theta}{dt^2} \]

So,

\[ I \frac{d^2 \theta}{dt^2} + mgd \theta = 0 \]

Same as simple harmonic motion.

\[ \frac{d^2 \theta}{dt^2} + \frac{mgd}{I} \theta = 0 \]

\[ k \rightarrow mgd \]

\[ m \rightarrow I \]
So,

\[ \omega = \sqrt{\frac{mgd}{I}} \]

\[ f = \frac{1}{2\pi} \sqrt{\frac{mgd}{I}} \]

\[ T = 2\pi \sqrt{\frac{I}{mgd}} \]

This is an easy way to measure \( I \) of an arbitrary shaped object.

- Find c.g. how?
- Hang it free to rotate around a point and measure \( T \).
\[ V = \frac{2S}{T} = \frac{2L \sin \theta}{T} = \frac{4L \sin \theta}{2\pi \sqrt{\frac{2L}{3g}}} \]

\[ = \frac{\sin \theta}{\pi} \frac{2L}{2L} \sqrt{\frac{3g}{2L}} \]

\[ = \frac{\sin \theta}{\pi} \sqrt{\frac{4L^2}{2L} \cdot 3g} = \frac{\sin \theta}{\pi} \sqrt{2L \cdot 3g} \]

For \( \theta = 30^\circ \)

\[ L = 1 \text{ m} \]

\[ = \frac{\sqrt{2}}{\pi} \sqrt{60} = \frac{1}{6} \cdot 7.5 \approx 1.1 \text{ m/s} = 2.5 \text{ MPH} \]

Not coincidentally that is \( \approx \) human walking speed. (unless you walk like a dork).

Note that \( V \propto \sqrt{L} \) and \( \sqrt{g} \)

\[ V \text{ on moon} = 2.5 \text{ MPH} \times \sqrt{\frac{1.7}{9.8}} = 1 \text{ MPH} \]
Another example, mostly to get practice.

A thin ring on a peg.

\[ d = R \]

\[ I = ? \]

Well, \( I = MR^2 \) for a thin walled cylinder, rotated about center. But we are rotating about the edge.

Recall Parallel-Axis Theorem

\[ I_P = I_{cm} + Md^2 \]

\( l \) \( \rightarrow \) distance from \( cm \) to \( p \).

So, \( I = 2MR^2 \)

\[ \Rightarrow \omega = \sqrt{\frac{dmg}{I}} = \sqrt{\frac{Rmg}{2MR^2}} = \sqrt{\frac{g}{2R}} \]

\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{R}{g}} \]

Twice the period of a simple mass \( R \) away.

Same as a mass at \( 2R \).
Torsional Spring.

\[ I = -K \theta \]

\[ = I \alpha = I \frac{d^2 \theta}{dt^2} \]

\[ \Rightarrow \frac{d^2 \theta}{dt^2} + \frac{K}{I} \theta = 0 \]

\[ \Rightarrow \text{SHM with } \omega = \sqrt{\frac{K}{I}} \]

\[ T = 2\pi \sqrt{\frac{I}{K}} \]

Can use this in a spring watch.