Today we will talk about "elasticity," the fact that objects are not really rigid.

This is another step on our path to realism. We have started with simple cases whenever possible, then add the complications bit by bit. E.g. Friction & Extended objects.

We will now deal with non-rigidity or non-in-extensibility.

We used "in-extensible" strings when solving pulley problems. Remember that that gave us a relation between the accelerations of the blocks.

But if we measured it carefully we would find that our calculation is not exactly matched.
For example, blocks sitting on table. If I remove the table, so the blocks can accelerate. I'd expect $a_y = 0$ if $m_1 = m_2$. $a_y < 0 \Leftrightarrow a_2 > 0$ if $m_1 > m_2$. Etc.

if $m_1 = m_2$

But I'd find that $a_1 < 0$ and $a_2 < 0$.

Remove the table, and both masses move down a little... because the string stretches.

How much does the string stretch?
Depends on the string: rubber band, wire, cable.

All materials actually stretch, some more than others.

Our goal today is to describe this stretching.

This is not fundamental, rather it is a parametric model that happens to work fairly well for most cases. Momentum conservation is fundamental... never seen to be violated. If it were that would be revolutionary. Our description here is an "effective model" that we know is incomplete, but is still useful to solve real world problems.
Stress = strain * Elastic Modulus, or \( \frac{\text{stress}}{\text{strain}} = \text{E. Mod.} \)

Called Hooke's law.

You may recognize that name from our discussion of springs. That is because it is the same thing.

"Stress" is a force, "strain" is a stretch distance.

But it is actually more general. We'll look at some types.

First, "Tensile stress" from tension in a string or wire, or other object.

\[
T = mg \\
\Delta \text{m}
\]

Find experimentally that string stretches, and for forces (masses) not too big, the stretch is proportional to the force, i.e., the tension.

**DEMO**

But two strings would halve the tension, so half the stretch. Three strings, etc.

So, in fact, cross-sectional area of string matters.
So,

"Tensile Stress" $= \frac{F_1}{A} = \text{Modulus \times Tensile Strain.}

"Tensile Strain" is $\frac{\Delta L}{L}$

"Modulus" is the name of the proportionality constant labelled $Y$ for Young's modulus.

$= \text{Property of a material. We don't need to calculate microscopic forces involved, just use this approx.}$

Units?

$\frac{F_1}{A} = Y \frac{\Delta L}{L}$

$\Rightarrow Y \text{ has units of force/area. } \frac{N}{m^2} \equiv \text{Pascal} = \text{Pa}$

$Y = \frac{F_1}{A} \frac{L}{\Delta L} \Rightarrow \text{small stretch, } \Delta L, \Rightarrow \text{large } Y.$

Note that although we used a stretching string as an example, this is valid also for compression.

Strings bend, but a steel bar stretches & compresses with $\approx$ the same $Y.$

So, how big is $Y$ for a steel bar?
Determine that experimentally.

I sit on a stool. The 4 legs compress.

\[
y = \frac{F}{A} \cdot \frac{L}{\Delta L} = \frac{100 \text{ kg} \times 10}{(1 \times 1 \text{ cm}^2) \times 4 \times 1 \text{ m}^2} \times \frac{1 \text{ m}}{10^{-3} \text{ m}} = \text{ an estimate upper limit.}
\]

\[\Rightarrow y = \frac{1000}{4/10000} \times 10^3 = \frac{10^7}{4} \times 10^{10} \text{ N/m}^2. \text{ Big.}
\]

Actually, the value is about \(2 \times 10^{11}\) even bigger!

Our estimate was indeed low. How much do the legs really compress?

\[
\frac{\Delta L}{L} = \frac{F/A}{2 \times 10^{11}} = \frac{1000/4/10000}{2 \times 10^{11}} = \frac{10^7}{8} \times 10^{11} = \frac{10^4}{8} \text{ m} = 0.01 \text{ mm}
\]

Steel is strong. If I did the same for a typical rope, it might stretch by a few percent ... cm/m.

A steel cable is stronger ... bigger \(y\).

\begin{itemize}
\item 1 Km
\item 1 mm
\end{itemize}

Copper has \(y = 1 \times 10^{11} \text{ N/m}^2\)

Suppose you hang a 10 gram (quarter) mass from it. How much does it stretch?

\[
\frac{F}{A} = y \Rightarrow \Delta L = \frac{L F}{AY} = \frac{10^3 (10^{-2} 10)}{(10^{-3})^2 10^{11}} = \frac{10^2}{10^5} = 10^{-3} \text{ m}
\]

\[
\frac{\Delta L}{L} = 1 \text{ mm}
\]
Of course the copper wire is far from massless, so this would really be far more complicated. But, an exercise nonetheless.

**BULK STRESS & STRAIN**

As you dive underwater, you can feel the water pressure pushing in on you... especially your ears. In fact, if you measured yourself you'd find that you shrink. This is a bulk stress & strain.

\[ dA \Rightarrow \text{there is a force acting on it} \]
\[ \text{inward, } dF/A \text{d}A = p \]
\[ \text{where } p \text{ is water pressure.} \]

Pressure \( = \frac{\text{Force}}{\text{Area}} \) units? Pa same as F/m²

This holds for air as well as water. Air pressure is \( \approx 10^5 \) Pa called 1 "atmosphere".

Pressure acts on all dA's in surface of object.

Shrinks by volume not length.

\[ \Rightarrow \frac{\Delta V}{V} \text{ is bulk strain.} \]
Again, \( \frac{\text{stress}}{\text{strain}} = \text{modulus} \).

\[ B = \frac{F/A}{\Delta V/V} = \frac{P}{\Delta V/V} \]

Actually, it is \( \Delta F \), so difference between internal and external pressures.

\[ B = \frac{-AP}{\Delta V/V} \]

Need the - to handle which is greater in \( \varepsilon \).

Again, \( B \) is a property of the material.

Also common to talk about "compressibility".

\[ K = \frac{1}{B} = \frac{\Delta V/V}{\Delta P} \]

Air is pretty easy to compress. Water is not. Nor steel.

E.g. water at bottom of ocean is only a couple \% denser.
We've dealt with $F_i$ so far. What about $F_{ii}$?

Do not rotate... "squish" or shear strain.

$$\text{Shear Modulus } = S = \frac{\text{Shear Stress}}{\text{Shear strain}} = \frac{F_{ii}/A}{x/h}$$

$$S = \frac{F_{ii}}{A} \frac{h}{x} = \frac{F_{ii}}{Ah}$$

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Elasticity & Plasticity

Things eventually snap under stress.

Only proportional for small strain.

Stress something enough and it starts to "give" more easily.

Breaking stress AKA tensile strength.

Can cause permanent deformation out here.

Elastic

Proportional

Snap
An example: A large mass hanging from two ropes.
Level now \(\Rightarrow\) which rope was originally shorter?

\[ T_1 + T_2 = mg \quad \text{and} \quad -T_1 L_1 + T_2 L_2 = 0 \]

\[ T_1 + T_1 \frac{L_1}{L_2} = mg \quad T_2 + T_2 \frac{L_2}{L_1} = mg \]

\[ T_1 = \frac{mg}{1 + \frac{L_1}{L_2}} \quad T_2 = \frac{mg}{1 + \frac{L_2}{L_1}} \]

\[ \Rightarrow T_2 > T_1 \]

So, rope 2 has more stress \(\Rightarrow\) more strain.
It should have been initially shorter.
If they were the same length, we could balance the stretch with larger area.
Now, some sample problems.

A yo-yo.

\[ \Sigma F = ma = T - mg \]

and

\[ \Sigma T = I \alpha = RT \]

Connect \( a \) and \( \alpha \):

\[ a = -\frac{a}{R} \]

\[ ma = T - mg \quad \text{and} \quad RT = -Ia/R \]

\[ = -\frac{a}{R} \left( \frac{1}{2}mR^2 \right) \]

\[ = -maR/2 \]

\[ \Rightarrow T = -\frac{1}{2}ma \]

\[ \Rightarrow ma = -\frac{1}{2}ma - mg \]

\[ \frac{3}{2}a = -g \quad a = -\frac{2}{3}g \quad \text{Checks?} \]
What if I used a more realistic yo-yo, string tied around a smaller radius peg (ignore its mass, but not its other effects).

Still have
\[ \sum F = T - mg = ma \]

But now the torque is different.

\[ \sum \tau = R_1 T = I \alpha = \frac{1}{2} m R_z^2 \left( - \frac{\alpha}{R_1} \right) \]

\[ \Rightarrow T = - \frac{1}{2} ma \left( \frac{R_z}{R_1} \right)^2 \]

\[ \Rightarrow ma = - \frac{1}{2} m \left( \frac{R_z}{R_1} \right)^2 \alpha - mg \]

\[ a \left[ 1 + \frac{1}{2} \left( \frac{R_z}{R_1} \right)^2 \right] = -g \]

\[ a = \frac{-g}{1 + \frac{1}{2} \left( \frac{R_z}{R_1} \right)^2} \] checks?

\[ R_1 \rightarrow 0 \Rightarrow \alpha \rightarrow 0 \text{ Does that make sense?} \]

(Maxwell's yo-yo demo).

Probably easier to make sense of it with an "energy" view.
\[ U_i + k_i = U_f + k_f \]

\[
mg \Delta h = \frac{1}{2} mv^2 + \frac{1}{2} I \omega^2
\]

\[
= \frac{1}{2} mv^2 + \frac{1}{2} \left( \frac{1}{2} m R_2^2 \right) \left( \frac{v}{R_1} \right)^2
\]

\[
g \Delta h = \frac{1}{2} V^2 \left[ 1 + \frac{1}{R_1} \left( \frac{R_2}{R_1} \right)^2 \right]
\]

\[
V^2 = \frac{2g \Delta h}{1 + \frac{1}{2} \left( \frac{R_2}{R_1} \right)^2}
\]

Small \( R_1 \) \( \Rightarrow \) Small \( V \)

because the \( \frac{1}{2} I \omega^2 \) term gets big. All the energy is in rotation rather than linear motion.
Let's use all this in a sample problem.

Door stop.

Swinging door has

\[ I = \frac{1}{3} ML^2 \]

angular velocity \( \omega \)

Hits door stop & stops. Why?

Because door stop exerts a force

\[ F_d \]

& hence \( \vec{F} = \dot{\vec{r}} \times \vec{F}_d \)

\[ \dot{\vec{L}} = -dF_d \]

At the same time, the hinge exerts a force.

It has two components.

What is \( F_h \)?

\[ F_h \]

What is \( F_h \)?

\[ \int T \, dt = \Delta L = L_f - L_i = -I \omega_0 \]

\[ -\int dF_d \, dt = -\int F_d \, dt = -I \omega_0 \Rightarrow \int F_d \, dt = \frac{I \omega_0}{\dot{\omega}} \]

Also, c.m. comes to rest.

\[ \Delta P = \int \Delta \vec{F} \, dt = \int -\left( \vec{F}_h + \vec{F}_d \right) \, dt = 0 - mV_{cm} \]

\[ \Rightarrow \int F_h + F_d \, dt = mV_{cm} = m \frac{UL}{2} \]
\[
\int (F_H + F_d) \, dt = \int F_H \, dt + \int F_d \, dt
\]

So

\[
m \omega_0 \frac{L}{2} = \int F_H \, dt + \frac{I \omega}{d}
\]

\[
\int F_H \, dt = \left( \frac{mL}{2} - \frac{I}{d} \right) \omega
\]

Again, I don't know \( F_H \), but I know its effect. Still useful. For example, suppose I want \( F_H = 0 \) to keep the door from ripping off its hinges?

\[
\Rightarrow \quad \frac{mL}{2} - \frac{I}{d} = 0
\]

or

\[
d = \frac{2I}{mL} = \frac{2}{3} \frac{mL^2}{mL} = \frac{2}{3} L
\]

For a uniform door, \( I = \frac{1}{3} mL^2 \)

Hit the right spot & there is no force.
Torque & linear momentum balance.

Same with a bat.