Statics: Calculate forces for objects in equilibrium.

This is pretty trivial for point particles. Just $\sum F = 0$.

For extended objects, it gets a little more interesting because even if $\sum F = 0$, the object can still rotate $\Rightarrow$ not in equilibrium.

What is equilibrium?

\~ means not accelerating

For most problems, you are also interested in "not moving" but that doesn't matter for $\sum \vec{F} = 0$, right?

What kinds of problems?

- Extended Object at rest, doesn't move or rotate
  a.g. Building -- don't fall
  Ladder -- don't slip
  Balanced beam -- balancers i.e. doesn't rotate
A trivial example:

Balance a uniform rod.

If it is to balance, then \( \ddot{x} = 0 \) and \( \ddot{y} = 0 \)

I.e., it doesn't accelerate.

\[
\ddot{x} = 0 \implies \sum F_x = 0 \implies \sum F_y = 0 \\
\sum F_z = 0
\]

What are the forces?

- Weight of rod
- Normal force from support

All in \( y \) direction, \( \pm x \) or \( \pm z \).

So, weight + normal = 0

Weight = \(?\) = mg

So, \( N = +mg \hat{y} \).

Of course, we know that is not sufficient. It can still pivot.
So, the second requirement for equilibrium is

\[ \ddot{\mathbf{r}} = 0 \Rightarrow \sum \dot{\mathbf{r}} = 0 \]

What is \( \dot{\mathbf{r}} \)?

\[ \dot{\mathbf{r}} = \mathbf{r}_i \times m_i \dot{g} \]

for each little \( m_i \).

Define origin at pivot

\[ r_i = r_i \] (\( x \))

Does sign make sense?

Of course, we have to sum up all the mass elements.

\[ \sum \dot{\mathbf{r}} = \sum \mathbf{r}_i \times m_i \dot{g} = \left( \sum m_i \mathbf{r}_i \right) \times \dot{g} \]

\[ = \sum m_i \mathbf{r}_i \times \dot{g} \]

\[ = \sum m_i \]

\[ = \frac{\sum m_i \mathbf{r}_i}{M} \times M \dot{g} \]

\[ = \frac{\sum m_i \mathbf{r}_i}{M} \times \dot{W} \]

Total torque is total weight at c.m.
Note that this is very similar to "center of mass" idea from long ago. It is slightly more general actually.

In this derivation, we assumed that $\hat{g}$ was the same for all points. That is reasonable on earth, but is not a good approximation always. For example, near a black hole, $V_{\text{grav}}(\vec{r})$ is a strong function of $\vec{r}$, can't approximate it as independent of $\vec{r}$ like on the surface of the earth.

In those cases, "center of gravity" $\neq$ "center of mass"
We won't worry about those cases.

So in our example,

$$\Delta \vec{v} = 0 \Rightarrow \vec{r}_{\text{cm}} \times M\hat{g} = 0 \Rightarrow \vec{r}_{\text{cm}} = 0$$

Pivot has to be at cm (cg) for no $\vec{v}$. Makes sense.

$\Rightarrow d = \frac{1}{2}$
A more interesting example is two pivots.

Again \[ \sum \vec{F} = 0 \Rightarrow N_1 + N_2 - Mg = 0 \]

\[ \sum \vec{E} = 0 \Rightarrow \vec{r}_1 \times \vec{N}_1 + \vec{r}_2 \times \vec{N}_2 = 0 \]

\[ d_1 N_1 \hat{z} + d_2 N_2 \hat{z} = 0 \]

(Signs are ok? If \( d < 0 \), then \( \hat{z} \) direction.)

If both \( d_1 \) and \( d_2 \) are same sign, then hopeless, since \( N_1 \), \( N_2 \) both positive, no solution.

So, \[ \Delta = d_1 - d_2 \]

but math is same.

\[ N_1 + N_2 - Mg = 0 \quad \frac{1}{2} d_1 N_1 + d_2 N_2 = 0 \]

\[ N_1 = -\frac{d_2}{d_1} N_2 \]

\[ (-\frac{d_2}{d_1} + 1) N_2 = Mg \]

\[ N_2 = \frac{Mg}{1 - \frac{d_2}{d_1}} \quad N_1 = \frac{Mg}{1 - \frac{d_1}{d_2}} \]

If both \( d_1, d_2 \) same sign, then either \( N_2 \) or \( N_1 \) < 0.
Notice that $N_1$ is not necessarily $= N_2$.

Only equal if $d_1 = -d_2$

An extreme example is

$$d_1 = -\frac{L}{2} \quad \text{and} \quad d_2 = 0$$

$$\Rightarrow N_1 = 0 \quad \Rightarrow N_2 = Mg$$

Less extreme

$$d_1 = -\frac{L}{4} \quad \text{and} \quad d_2 = \frac{L}{2}$$

$$N_1 = \frac{\frac{L}{2} Mg}{\frac{L}{2} - (-\frac{L}{4})} = \frac{4\frac{1}{3}}{2} Mg = \frac{2}{3} Mg$$

$$N_1 = \frac{-\frac{L}{4} Mg}{-\frac{L}{4} - \frac{L}{2}} = \frac{4\frac{1}{3}}{3} Mg = \frac{1}{3} Mg$$

We calculated the torque around an axis through the cm. That torque $= 0 \Rightarrow$ no $\vec{\tau}$ (tendency to rotate).

Of course we won't want rotation about any axis.

So, the torque should be zero about all axes.

I used the cm because it is easiest. Could have used any other.
Suppose I had used an axis through one of the pivots, say pivot 2.

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A less trivial example

A rod of length $L$ and mass $M$
supports a block of mass $m$,
hanging at distance $d$ from the center of the rod.
What is the tension in the ropes and what is $\theta$?

\[ \sum F = 0 \Rightarrow T_1 \sin \theta_1 + T_2 \sin \theta_2 - m_1 g - mg = 0 \text{ in y} \]
\[ \sum \vec{t} = 0 \Rightarrow \text{Choose c.m. of rod.} \]

\[-\frac{1}{2} T_1 \sin \theta_1 + \frac{1}{2} T_2 \sin \theta_2 + 0 \text{mg} - d m \text{,} \quad \text{mg} = 0 \]

\[ \Rightarrow T_2 \sin \theta_2 = T_1 \sin \theta_1 = 2d m_1 g / L \]

\[ T_2 \sin \theta_2 - [m_1 g + mg - T_2 \sin \theta_2] = 2d m_1 g / L \]

\[ 2 T_2 \sin \theta_2 = \frac{2d m_1 g}{L} + m_1 g + mg \]

\[ T_2 = g \left[ \frac{d}{L} m_1 + \frac{m_1 g}{2} \right] / \sin \theta_2 \]

Check: Units = mass times $g$. OK.

$\theta_2 \to 0 \Rightarrow T_2 \to \infty$ OK

d $\to 0 \Rightarrow T_2 \to g (m + m_1) / 2$ OK

$m'_s \to 0 \Rightarrow T \to 0$. 

We can plug in to find $T_1$. But first, what do we expect by symmetry?

$$d \to -d \Rightarrow T_2 \to T_1 \quad \frac{\Theta_2}{2} \to \Theta_1$$

So, I expect

$$T_1 = g \left[ \frac{m_1 + m_1}{2} - \frac{d m_1}{L} \right] \sin \Theta_1$$

Doing the algebra:

$$T_1 \sin \Theta_1 = m_1 g + m_2 g - T_2 \sin \Theta_2$$

$$= m_1 g + m_2 g - g \left[ \frac{d}{L} m_1 + \frac{m_1 + m_1}{2} \right] \sin \Theta_2 \sin \Theta_2$$

$$= \left[ \frac{m_1 + m_1}{2} - \frac{d m_1}{L} \right]$$

Finally, $\Sigma F_x = 0$ gives us a relation between the $\theta_i$:

$$\cos \Theta_1 = \frac{T_2 \cos \Theta_2}{T_1}$$
What tension is required in the rope to make hinge A have no horizontal force?

YF gives a bunch of numbers but I'll just solve symbolically; plugging in the numbers is pretty mindless.

As usual, we want \( \sum \vec{F} = 0 \) and \( \sum \vec{E} = 0 \). I'll separately write the horizontal and vertical components of the forces on the hinges: \( A_x, A_y \) and \( B_x, B_y \). We want \( A_x = 0 \), but we'll carry it along for a little while at least.

(1) \( \sum F = 0 \Rightarrow \sum F_x = 0 \Rightarrow A_x + B_x - T \cos \Theta = 0 \)
(2) \( \sum F = 0 \Rightarrow A_y + B_y - mg + T \sin \Theta = 0 \)

\( \sum \vec{E} = 0 \) I'll calculate the torques about the center of gravity of the gate.

\[ 0 = -A_y \frac{L}{2} - B_y \frac{L}{2} - A_x \frac{h}{2} + B_x \frac{h}{2} + T \sin \Theta \frac{L}{2} + T \cos \Theta \frac{h}{2} \]

(3) \[ A_y l + B_y l + A_x h - B_x h = l T \sin \Theta + h T \cos \Theta \]
A couple asides. First, what about the angles between the forces and \( \hat{r} \). I've simplified this by just using the two components. So, for \( A_x \)

\[
\vec{r} = \vec{r} \times \hat{A}_x =
= \vec{r} \left| \vec{A}_x \right| \sin \beta
= \left| \vec{r} \right| \left| \vec{A}_x \right| \sin \beta
\sin \beta = \frac{h}{2 \sqrt{1 - \frac{h^2}{2}}} \\
\Rightarrow \left| \vec{r} \right| \sin \beta = \frac{h}{2}
\]

Rather than carrying these angles around, it is easier to just use the normal distance

\[
\Rightarrow \frac{h}{2}
\]

Second, why didn't I factor out the \( T \) to get \( T (\sin \theta + \cos \theta) \)? Because \( T \sin \theta \) and \( T \cos \theta \) are forces present in equations (1) and (2). I'll wait till I put it all together and then pull out the trig.
A circular disk 0.500 m in diameter, pivoted about a horizontal axis through its center, has a cord wrapped around its rim. The cord passes over a frictionless pulley \( P \) and is attached to an object that weighs 240 N. A uniform rod 2.00 m long is fastened to the disk, with one end at the center of the disk. The apparatus is in equilibrium, with the rod horizontal (Fig. 11–38).

a) What is the weight of the rod? b) What is the new equilibrium direction of the rod when a second object weighing 20.0 N is suspended from the other end of the rod, as shown by the broken line? That is, what angle does the rod then make with the horizontal?

\[
\sum F = 0 \Rightarrow \sum F_y = 0
\]

\[
T = m_1 g
\]

\[
\sum \vec{F} = 0 \Rightarrow RT - \frac{1}{2} m_2 g = 0
\]

\[
l = \frac{2RT}{L} = \frac{2R m_1 g}{L} = \text{Units} = 0.5 \times 240 = 60 \, \text{N}
\]
Then we add a weight. What angle is equilibrium?

\[ T = m_a g \frac{L}{2} \sin(90 - \theta) + m_3 g \frac{L}{2} \sin(90 - \theta) \]

\[ \Rightarrow \sin(90 - \theta) = \cos \theta = \frac{R \cdot m_1 g}{m_2 g \frac{L}{2} + m_3 g L} \]

\[ \cos \theta = \frac{2m_1 R}{(m_2 + 2m_3) L} \quad \text{(Unit less)} \]

\[ \theta = \cos^{-1} \left[ \frac{0.5 \cdot 240}{(60 + 40)^2} \right] = \cos^{-1} \left[ \frac{120}{200} \right] = \cos^{-1} \left( \frac{3}{5} \right) = 53.1^\circ \]

Note that it won't stop there... it is just that this point is an equilibrium point.
In Fig. 11-36 a 6.00-m-long uniform beam is hanging from a point 1.00 m to the right of its center. The beam weighs 140 N and makes an angle of 30.0° with the vertical. At the right-hand end of the beam a 100-N weight is hung; an unknown weight \( w \) hangs at the other end. If the system is in equilibrium, what is \( w \)? Ignore the thickness of the beam.

b) If the beam makes instead an angle of 45.0° with the vertical, what is \( w \)?

\[
\Delta F = 0 \Rightarrow -W - m_1 g + T - m_2 g = 0
\]

Not particularly useful. Just tells us the tension which we don't need in this case.

\[ \sum \tau = 0 \text{ about all points. Pick a point. If we use the pivot point (where the cable is attached) then we don't need to know } T. \text{ Sounds good. Other options valid, except for axis through point where } \text{W is connected. Why?} \]

So,

\[ 0 = (\frac{L}{2} + d) W \sin \theta + dm_1 g \sin \theta - (\frac{L}{2} - d) m_2 g \sin \theta \]

\[ \sin (180 - \theta) = -\sin \theta \]

or

\[ - (r mg \sin \theta) \]

So \( \sin \theta \) cancels. Make sense?

Could think of components:

\[ W = \frac{(\frac{L}{2} - d) m_2 g - dm_1 g}{(\frac{L}{2} + d)} \]
\[ W = \frac{(L - 2d)m_2g - 2d m_1g}{L + 2d} \]

I won't pull out the g's because we are given weights not masses.

\[ W = \frac{L m_2g - 2d (m_1g + m_2g)}{L + 2d} \]

Units: OK, \( \lim_{d \to 0} d \to 0 \Rightarrow W = m_2g \) OK.

\[ \lim_{q \to 0} q \to 0 \Rightarrow W \to 0 \] OK.

If \( m_2 \to 0 \Rightarrow W < 0 \) I have to push up.

If \( w = 0 \), then

\[ (L - 2d)m_2g = 2d m_1g \]

\[ \Rightarrow \frac{L - 2d}{2} m_2g = \frac{d}{2} m_1g \Rightarrow \text{balance. OK.} \]

Fine, so we can plug in.

\[ W = \frac{6 \cdot 100 - 2 \cdot 1 \cdot (140 + 100)}{8} = \frac{600 - 480}{8} \]

\[ = \frac{120}{8} = \frac{30}{2} = 15 \text{ N} \]
Two friends are carrying a 200-kg crate up a flight of stairs. The crate is 1.25 m long and 0.50 m high, and its center of gravity is at its center. The stairs make a 45.0° angle with respect to the floor. The crate also is carried at a 45.0° angle, so that its bottom side is parallel to the slope of the stairs (Fig. 11-40). If the force each person applies is vertical, what is the magnitude of each of these forces? Is it best to be the person above or below on the stairs?

\[ \sum F = 0 \Rightarrow F_1 + F_2 - mg = 0 \Rightarrow F_1 = mg - F_2 \]
\[ F_2 = mg - F_1 \]

\[ \sum \tau = 0 \quad \text{Let's pivot about c.m.} \]
\[ F_1 d \sin \theta_1 = F_2 d \sin \theta_2 \]
\[ d' \text{'s cancel} \]

\[ \theta_1 = 90 - \alpha - \psi \]
\[ \theta_2 = 90 - \alpha + \psi \]

\[ \sin \theta_1 = \sin (90 - \alpha - \psi) = \cos (\alpha + \psi) = \cos \alpha \cos \psi - \sin \alpha \sin \psi \]
\[ \sin \theta_2 = \sin (90 - \alpha + \psi) = \cos (\alpha - \psi) = \cos \alpha \cos \psi + \sin \alpha \sin \psi \]

\[ \Rightarrow F_1 \sin \theta_1 = F_2 \sin \theta_2 = (mg - F_1) \sin \theta_2 \]

\[ F_1 = \frac{mg \sin \theta_2}{\sin \theta_1 + \sin \theta_2} = mg \frac{\cos \alpha \cos \psi + \sin \alpha \sin \psi}{\cos \alpha \cos \psi - \sin \alpha \sin \psi} \]
\[ = mg \frac{\cos \alpha \cos \psi + \sin \alpha \sin \psi}{\cos \alpha \cos \psi \left(1 + \tan \alpha \tan \psi\right)} \]
\[ = \frac{1}{2} mg \left(1 + \tan \alpha \tan \psi\right) \]
We are given $\alpha$ but not $\psi$. How to find $\psi$?

\[ \tan \psi = \frac{h/2}{L/2} = \frac{h}{L} \]

\[ F_1 = \frac{1}{2} mg \left(1 + \frac{h/2}{L/2} \tan \alpha \right) \]

Plug in, \[ F_1 = \frac{1}{2} \left(1 + \frac{0.5}{0.25} \cdot \tan 45 \right) mg \]

\[ = \frac{1}{2} \left(1 + \frac{2}{1} \cdot 1 \right) mg \]

\[ = \frac{1}{2} \left(1 + \frac{2}{1} \right) mg \]

\[ = \frac{3}{2} mg \]

\[ \Rightarrow F_2 = mg - \frac{3}{10} mg = \frac{3}{10} mg \] smaller.