Review:

$$\Delta \vec{p} = m \vec{a} \quad \Rightarrow \quad \Delta \vec{v} = \vec{F}$$

$$= \frac{d\vec{p}}{dt} = \frac{d\vec{v}}{dt}$$

$$\vec{p} = m\vec{v}$$

$$\vec{l} = I \vec{\omega}$$

Vectors are a bit tricky, particularly in the gyroscope.

So, another demo or two.

Bicycle wheel on rotatable stool

\[ \vec{r} \] and \( \vec{l} \) point up

Now rotate it to \( \bigcirc \) what happens?

A bit tricky because \( \vec{l} \) is only "conserved" along the z-axis. That is because the freedom to rotate about \( \hat{z} \) makes that a closed system. But \( \hat{x} \) and \( \hat{y} \) are tied to the earth, so not "closed" there.

\( \Rightarrow \) \( L_z \) conserved
Initially

$$\uparrow I_0 \omega_0 \quad \Rightarrow \quad \downarrow I_0 \omega_0$$

Finally

$$\uparrow I_0 \omega_0 \quad \Rightarrow \quad \downarrow I_0 \omega_0$$

$$L_z = I_0 \omega_0 \cos \theta$$

We have to add the missing $L_z$ by rotating.

So person will rotate to the left.

If all 3 dimensions were free, we'd tilt and rotate.

so horizontal components cancel.
Gyro suitcase demo

![Diagram of C]

So \( \vec{l} \) points out of the page.

If I try to turn it \( \vec{l} \), I must apply a torque pointing up. That changes \( \vec{l} \) (\( \vec{\tau} = \frac{d\vec{l}}{dt} \)) and makes \( \vec{l} \) point more up.

Check directions a few times.

This is exactly the same as the bicycle wheel gyroscope.

\[
\vec{\tau} = \frac{d\vec{l}}{dt}
\]

if \( \vec{l} \) does not change, which can seem counter-intuitive.
Now, some sample problems.

A yo-yo.

\[ \sum F = ma = T - mg \]

and

\[ \sum T = I \alpha = RT \]

So,

\[ \alpha = -\frac{a}{R} \]

Connect \( a \) and \( \alpha \)

\[ ma = T - mg \quad \text{and} \quad RT = -Ia/R \]

\[ = -\frac{a}{R} \left( \frac{1}{2}mR^2 \right) \]

\[ = -maR/2 \]

\[ \Rightarrow T = -\frac{1}{2}ma \]

\[ \Rightarrow ma = -\frac{1}{2}ma - mg \]

\[ \frac{3}{2}a = -g \quad a = -\frac{2}{3}g \quad \text{Checks?} \]
What if I used a more realistic yo-yo, string tied around a smaller radius peg. (Ignore its mass, but not its other effects).

Still have
\[ \Sigma F = T - mg = ma \]

But now the torque is different.

\[ \Sigma T = R_1 T = I \alpha = \frac{1}{2} m R_z^2 \left( -\frac{\alpha}{R_1} \right) \]

\[ \Rightarrow T = -\frac{1}{2} ma \left( \frac{R_z^2}{R_1} \right) \]

\[ \Rightarrow ma = -\frac{1}{2} m \left( \frac{R_z}{R_1} \right)^2 \alpha - mg \]

\[ \alpha \left[ 1 + \frac{1}{2} \left( \frac{R_z}{R_1} \right)^2 \right] = -g \]

\[ \alpha = \frac{-g}{1 + \frac{1}{2} \left( \frac{R_z}{R_1} \right)^2} \]

Checks?

\[ R_1 \rightarrow 0 \Rightarrow \alpha \rightarrow 0 \] Does that make sense?

(Maxwell's yo-yo demo).

Probably easier to make sense of it with an "energy" view.
\[ U_i + k_i = U_f + k_f \]
\[ mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \]
\[ = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2} MR_i^2\right)\left(\frac{v}{R_i}\right)^2 \]
\[ gh = \frac{1}{2}v^2\left[1 + \frac{1}{2}\left(\frac{R_i}{R_i}\right)^2\right] \]
\[ v^2 = \frac{2gh}{1 + \frac{1}{2}\left(\frac{R_i}{R_i}\right)^2} \]

Small \( R_i \Rightarrow \) Small \( v \)

because the \( \frac{1}{2}I\omega^2 \) term gets big.

All the energy is in rotation rather than linear motion.
Interlude: Calculate $I$ for disk rotated about its center.

\[
I = \frac{1}{2} m r_e^2 = \int dm r^2
\]

Take infinitesimally thin rings and add them up.

What is the mass of each ring?

\[
dm = \rho \, dA
\]

where $\rho$ is mass density per area.

\[
\rho = \frac{M}{\pi R^2}
\]

$dA$ is the infinitesimal area of the ring at $r$.

\[
dA = 2\pi r \, dr
\]

So,

\[
I = \int dm \, r^2 = \int_0^R \rho (2\pi r \, dr) \, r^2
\]

\[
= \frac{M}{\pi R^2} (2\pi) \int_0^R r^3 \, dr
\]

\[
= \frac{2M}{R^2} \left[ \frac{r^4}{4} \right]_0^R = \frac{2M}{4R^2} \cdot R^4
\]

\[
= \frac{1}{2} MR^2
\]
Calculating angular acceleration. This is a messy example -- the mess is intentional to show how focussing on one piece at a time lets you simplify the mess.

Pulley 1
= uniform disk
  $m_1$, $R_1$

Pulley 2
= hollow disk
  $m_2$, $R_2$

Solve by applying Newton's 2nd law to each object. (A, B, 1, 2).
For the pulleys, N and is $\Delta T$, not just $\Delta F$.

$A \Rightarrow \sum F_A = T_1 - m_A g = m_A \alpha_A \quad (1)$

$P_1 \Rightarrow$ Rope holding $P_1$ up has tension = $m_1 g$.
(That is $\sum F$, but it ain't useful).

Also, $\sum T_i = I_1 \alpha_1 = R_1 T_1 - R_1 T_2$

$\Rightarrow R_1 (T_1 - T_2) = \left( \frac{1}{2} m_1 R_1^2 \right) \left( -\frac{\alpha_1}{R_1} \right)$
\[ T_1 - T_2 = - \frac{1}{2} m_1 \alpha_A \]  

\[ P_2 \Rightarrow \sum T_2 = R_a T_2 - R_2 T_3 = I_2 \alpha_2 \]

\[ R_2 (T_2 - T_3) = m_2 R_a^2 \left( -\alpha_A / R_2 \right) \]

\[ T_2 - T_3 = -m_2 \alpha_A \quad \text{[No} \frac{1}{2} \text{ because the pulley is hollow disk.]} \]

\[ B \Rightarrow \sum F = T_3 - m_B g = m_B \alpha_B \]

\[ T_3 - m_B g = m_B (-\alpha_A) \]  

4 objects \( \Rightarrow \) 4 equations \( \neq \) 4 unknowns. (implicitly used \( \alpha_A, \alpha_1, \alpha_2, \alpha_B \) relations).