Suppose I have two particles interacting with each other, but nothing else (a simple view of the universe).

By Newton’s 3rd law, every force has an equal and opposite reaction force.

So, if particle 2 exerts a force \( \vec{F}_2 \) on particle 1, then particle 1 exerts a force \( \vec{F}_1 = -\vec{F}_2 \) on particle 2.

If there are no other forces (isolated system), then

\[
\vec{F}_1 = \frac{d\vec{p}_1}{dt}
\]

since \( \vec{F}_1 = \vec{F}_{\text{net}} \) on particle 1.

\[
\vec{F}_2 = \frac{d\vec{p}_2}{dt} = -\vec{F}_1 = -\frac{d\vec{p}_1}{dt}
\]

\[
\Rightarrow \frac{d\vec{p}_2}{dt} + \frac{d\vec{p}_1}{dt} = 0
\]

\[
\Rightarrow \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0
\]

\[
\Rightarrow \vec{p}_1 + \vec{p}_2 \text{ is a constant.}
\]

\[
\Rightarrow \text{Conservation of momentum is similar to energy.}
\]

Actually called conservation of linear momentum. We’ll deal with “rotational momentum” later.
This holds if there are more than two particles as well.

\[ \sum_i \vec{p}_i \text{ initial} = \sum_i \vec{p}_i \text{ final} \]

Since this is a vector equation, it is really 3 independent equations.

Momentum is conserved in each direction.

\[ \sum_i p^i_x = \sum p^f_x \]
\[ \sum p^i_y = \sum p^f_y \]
\[ \sum p^i_z = \sum p^f_z \]

What is important, though, is that this is only true for an isolated system.

= No external forces

= \( \sum \vec{F} = 0 \)

If there are other forces, then they introduce a \( \frac{d\vec{p}}{dt} \) component that we would have to include to get momentum conservation.
As the dumbbell accelerates right, down the ramp, the skateboard accelerates left.
Sample Problem: Momentum Conservation in a pitching machine.

If the pitching machine launches a ball of mass $m$ with speed $v$, then it will recoil with what velocity?

\[
Mv + mv = 0
\]

\[
v = -\frac{m}{M}v
\]

If it were on a frictionless surface, then it would slide away. Not isolated.

What if it were not frictionless? Suppose $\mu_s$ is large enough that it doesn't move. Is momentum conserved?

Yes, the earth recoils, but $v$ is very small because $M$ is so big.

What happens when the batter hits the ball? Momentum is again conserved.

Before
\[
\vec{m}v_i
\]

After
\[
\vec{m}v_i + \vec{m}v_i = \vec{m}v_j + M\vec{V}_f
\]
Is momentum conserved during the baseball's flight?

Even if \( |\vec{v}_i| = |\vec{v}_f| \), the direction is different, so the momentum, \( \vec{p} \), has to have changed. The y component (vertical) changed from \( v_{y1} \) to \( -v_{y2} \).

So, we lost \( 2m v_{y1} \) somehow. Where did it go?

During that time, the earth "falls" up toward the ball (Newton's third law). The speed of the earth increases by \( \frac{1}{2} \frac{\Delta m v_{y1}}{M_E} \) during the ball's flight.

This is of course too small to be measured for something as big as the earth.

Note that the earth gets a \( -\frac{\Delta m v_{y1}}{M_E} \) velocity change when the ball is first hit.

There is another \( -\frac{\Delta m v_{y1}}{M_E} \) change when the ball hits the ground.

We really need to include the earth even in the ball-bat interaction to see the full momentum conservation.

...although we'll come back to this.
Sample Problem: Bullet fired from a rifle into a block.

\[ m_1 \dot{V}_1 + m_b \dot{V}_b = 0 \]

\[ \Rightarrow \dot{V}_i = -\frac{m_b}{m_1} \dot{V}_b \]

\[ m_b \dot{V}_b = (m_b + M_B) \dot{V}_B \]

Recoil of rifle is conservation of momentum.
Shooter gets same momentum change as target (in fact more if you include the effects of air drag).
It doesn’t “hurt as much” because it is spread over a large area, and a longer time, and has less kinetic energy. Same \( mV \), but less \( mV^2 \).

How high does the block go?

By work-energy theorem, or conservation of energy,

\[ mgh = \frac{1}{2} mV^2 \]

\[ (m_b + m_B)gh = \frac{1}{2} (m_b + m_B) V_B^2 = \frac{1}{2} (m_b + m_B) \left( \frac{m_b \dot{V}_b}{m_b + m_B} \right)^2 \]

\[ V_b = \sqrt{2gh \left( \frac{m_b + m_B}{m_b} \right)} \]

So, we can calculate things from conservation of momentum even without knowing the forces.
We said about Newton's 3rd law that each force has an equal reaction force. So, if a big astronaut pushes a little astronaut, the reaction force is equal.

\[ F \rightarrow 0 \rightarrow F \]

\[ \vec{a}_B = M \vec{a}_L = m \vec{a}_L \]

If I knew the details of the force, I could integrate it with time, and get motion of each.

Even without knowing anything about the force, I know that

\[ M \vec{V}_B = -m \vec{V}_L \]

This turns out to be very useful. For example, when a proton and anti-proton interact, the forces can be complex... and the time scale is very small. I could never integrate \( F(t) \) to find the results.

But conservation of energy & momentum allows me to calculate a lot about the interaction.

We'll come back to that, but first, I want to talk about impulse.
We introduced momentum with

\[ \vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \Delta \vec{p} = \int_{t_i}^{t_f} \vec{F} \, dt = \vec{J} \]

This is called impulse.

Impulse is change in momentum.

\[ \Delta \vec{p} \] is area under the force-time curve.

Again, the word impulse has a different meaning in common language. It usually implies a short time.

Impulse need not be just over a short time.

A rocket burning all its fuel in a quick blast or slowly over a long time could have the same \( \Delta \vec{p} \). = Impulse.

\[ \begin{align*}
\text{F} & \quad \text{F} \\
\text{v.s.} & \quad \text{v.s.} \\
\text{t} & \quad \text{t}
\end{align*} \]

Same impulse because same \( \Delta \vec{p} \). = same \( \int F \, dt \).

Of course, there is a difference between these two scenarios. For example, the one on the left could kill the occupants, because the peak force is jarringly large.
Although Impulse is just as appropriate for long or short time scales we usually talk about impulsive forces being those which act over a short time period. Why?

Because in that case (short time period) the force is very large and all other forces can be neglected. This motivates that jargon.

Equal impulse but different peak force is useful. E.g.

**Bow & Arrow** --- slowly stretch a spring, then \( F t \) when released causes \( A P \).

When the arrows hits, \( A P \) is the same but over a much shorter time.

Same impulse, but large peak force.

**A hammer is the same idea...** small force over a long time yields huge force over a short time.

Which has more impulse, a big or small hammer?

It is useful to think about ways to do the opposite... take a very large force and spread it over a longer time. The total impulse doesn't change, but the peak force does.

1). Airbag

2). Boxing gloves

3). Bending your legs when you fall, or rolling.

4). Crumple zones in a car

**Others?** Bike helmet, foot pads e.g. your heel.

**Tennis ball vs. golf ball.**

**Questions on impulse?**
Collisions between objects

Conservation of momentum allows us to analyze collisions between objects fairly easily.

Even without knowing the forces, we can use conservation of momentum to determine something about final velocities.

But, momentum conservation only holds in a closed system... otherwise momentum could be transferred out of the system.

⇒ Still conserved, but within a larger system.

The falling baseball is a good example. Momentum conservation is clear only when we take the earth into account.

Fortunately, we can approximate most collisions as a closed system because during the collision the force between the two objects is huge compared to other forces.

For example, I can ignore gravity when considering the force between a baseball and bat. Over the millisecond or so that the bat exerts its force, it causes a large impulse, ΔP, but gravity causes a negligibly small one.

\[
\Delta P = \int F_x \text{dt} = \int (F_+ - mg) \text{dt}
\]

So, momentum is conserved in any type of collision.
Collisions in one-dimension

\[ m_1 v_{i_1} \rightarrow \quad m_2 v_{i_2} \quad \leftarrow 0 \]

What happens when they collide? Momentum is conserved.

\[ m_1 v_{i_1} + m_2 v_{i_2} = m_1 v_{f_1} + m_2 v_{f_2} \]

\[ V_{f_1} = \frac{m_1 v_{i_1} + m_2 v_{i_2} - m_2 v_{f_2}}{m_1} \]

What if \( m_1 v_{i_1} = -m_2 v_{i_2} \)?

\[ m_1 v_{f_1} = -m_2 v_{f_2} \]

They bounce back, and if I measure \( v_{2f} \), I know \( v_{1f} \).

Can I predict \( v_{2f} \) somehow? Using something else I know?

Conservation of energy,

\[ \frac{1}{2} m_1 v_{i_1}^2 + \frac{1}{2} m_2 v_{i_2}^2 = \frac{1}{2} m_1 v_{f_1}^2 + \frac{1}{2} m_2 v_{f_2}^2 \]

\[ \Rightarrow 2 \text{ equations and 2 unknowns (assuming I know the masses and initial velocities).} \]

\[ V_{f_1} = \left( \frac{m_i - m_2}{m_1 + m_2} \right) v_{i_1} + \left( \frac{2m_2}{m_1 + m_2} \right) v_{i_2} \]

\[ V_{2f} = \left( \frac{2m_1}{m_1 + m_2} \right) v_{i_1} + \left( \frac{m_2 - m_1}{m_1 + m_2} \right) v_{i_2} \]
Suppose you have two identical balls (like billiards) and one is at rest.

\[ V_{1f} = \frac{m_1 - m_2}{m_1 + m_2} V_{1i} + \left( \frac{2m_2}{m_1 + m_2} \right) V_{2i} \]

\[ L = 0 \]

\[ \Rightarrow V_{1f} = 0 \]

\[ V_{2f} = \frac{2m_1}{m_1 + m_2} V_{1i} + \frac{m_2 - m_1}{m_1 + m_2} V_{2i} \]

\[ L = \frac{2m}{2m} \]

\[ \Rightarrow V_{2f} = V_{1i} \]

1st ball stops & 2nd continues with its speed.

See this in billiards or the old toy.

\[ \text{Diagram:} \]
Suppose \( m_1 = m_2 = m \) and \( V_{1i} = -V_{2i} \).

\[ V_{1f} = V_{2i} \]

\[ V_{2f} = V_{1i} \]

Bounce back with same speed.

You also can see this with the old ball bearing toy.

BUT, there is a problem with this, if I used clay instead of steel ball-bearings, they wouldn’t bounce back.

Raw eggs are probably a better example.

Why? What is wrong?

Kinetic energy is mechanical energy.

So my "energy conservation" is only true if the forces involved are conservative (i.e. mechanical energy is conserved).

Total energy is conserved always. But if a lot of the energy goes into sound or heat or deforming the objects, then kinetic energy is not conserved.

Energy is conserved, just not kinetic energy.
So, just as we had two "classifications" for forces, conservative and non-conservative, or conserving mechanical energy or not, we also have two types of collisions.

Elastic Collisions = Kinetic energy is conserved
Inelastic Collisions = Kinetic energy is not conserved

The force classes were pretty clear cut,
\[ \int_{a}^{b} F(x) \, dx = - \int_{b}^{a} F(x) \, dx \quad \text{or it does not.} \]

Elastic or inelastic is more "mushy."

- Purely elastic is rare -- not impossible but rare.
  - For example, the billiards or ball bearing give off sound energy.
  - Scattering of two planets could have tidal energy changes
  - Etc.

So, they are "nearly elastic," but not purely.

The only purely elastic collisions I can think of are interactions of fundamental particles. There, the only internal energy is mass.

On the other extreme, if two lumps of clay stick together all the kinetic energy is lost to internal energy. \( P \) still conserved!

Reality is usually between... but we can talk about elastic collisions as a good approximation for many things.

\( P \) is always conserved. \( K \) is conserved only in elastic collisions.
Each pair of balls include one elastic, one inelastic. When dropped on the floor they show the difference between elastic and inelastic collisions.
What about collisions in $>1$ dimension?

A simple example is two billiard balls.

Before: $\vec{m}_1 \vec{V}_1$  \hspace{1cm} $\vec{M}_2$, $\vec{V}_2 = 0$

After: ?

Conservation of momentum tells us

$$\sum \vec{p}_i = \sum \vec{p}_f$$

$$\frac{1}{2} m_1 V_{1x} + m_2 V_{2x} = m_1 V_{1xf} + m_2 V_{2xf}$$

$$\frac{1}{2} m_1 V_{1y} + m_2 V_{2y} = m_1 V_{1yf} + m_2 V_{2yf}$$

$$m_1 |\vec{V}_1| + 0 = m_1 |\vec{V}_{1f}| \cos \alpha_1 + m_2 |\vec{V}_{2f}| \cos \alpha_2$$

$$0 + 0 = m_1 |\vec{V}_{1f}| \sin \alpha_1 + m_2 |\vec{V}_{2f}| \sin \alpha_2$$

2 equations $\div$ 4 unknowns. Give me $|\vec{V}_{1f}|$ and $\alpha_1$, and I can find the rest.

If elastic, I have one more equation, from

$$\frac{1}{2} m_1 V_{1}^2 = \frac{1}{2} m_1 V_{1f}^2 + \frac{1}{2} m_2 V_{2f}^2$$

Comment on predictability of point particle interactions vs. an extended object like a billiard ball.
Sample Problem  
Bullet through a block

\[ \begin{align*}
\vec{v}_t & \rightarrow \text{block} \\
& \rightarrow \text{spring} \\
& \rightarrow \text{displacement} \\
& \rightarrow \Delta v_f \\
\end{align*} \]

Given \( m's, k, v_i, \) find \( v_f. \)

\[ m_b v_i = m_b v_f + m_B v \quad \text{velocity imparted to block.} \]

\[ v = \frac{m_b (v_i - v_f)}{m_B} \quad \text{or} \quad v_f = \frac{m_b v_i - m_B v}{m_b} \]

\[ \frac{1}{2} m_B v_f^2 = \frac{1}{2} k d^2 \quad \text{or} \quad v_f = \frac{m_b v_i - m_B \frac{k}{m_B} d}{m_b} \]

If \( v_f = 0, \) absorbed,

\[ m_b v_i = m_b \frac{k}{m_B} d = \sqrt{k m_B} d \]

\[ v_i = \sqrt{\frac{k m_B}{m_b}} d = \sqrt{\frac{\frac{\text{kg} \cdot \text{m}}{\text{m} \cdot \text{s}^2}}{\frac{\text{kg}}{\text{m}}} \cdot \frac{\text{kg}}{\text{m} \cdot \text{s}^2} \cdot \frac{\text{m}}{\text{s}} \cdot \text{m} \cdot \text{kg}}{\text{kg}} \]

\[ m = \frac{m}{s} \text{ or } k \]
1). Bounce a ball. Assume it is elastic... not quite but close.

- How much momentum does the ball transfer to the earth during the collision?

\[ 2mv \]

\[ \rightarrow \text{ twice because } v_f = -v_i \] (See above)

\[ m, v_i + M \cdot 0 = +m, v_f + MV \]

\[ MV = 2m, v_i \]

Earth gets a double whammy... and winces in pain.

---

Demo 24.07

Stick the red clay to the side of the metal pendulum bob facing the block of wood. Pull the bob back to the upright metal rod, then release it. When the bob strikes the block of wood, the block teeters, but does not tip over. Replace the clay with a rubber foot pad and repeat the procedure. The block of wood easily tips over.
- Where does that momentum end up?

\[ \Delta p \text{ of earth and ball while moving up & down.} \]

Impact pushes earth down.
Gravitational attraction to ball pulls it back.
"A pretty unhealthy relationship."

Note that I could just as easily talked about this from the earth's point of view.

2) A related example. Two balls. Basketball & tennis ball.

What will happen when I drop them together, like this \\( \bigcirc \)?

\[ m_B \text{ & } m_c \text{ collide, then } m_B \text{ & } m_T \text{ collide.} \]

What is the final speed of the tennis ball?

1st they both have the same speed when they reach the ground, right.

\[ v = ? \]

\[ v = v_0 - g t \]

\[ t = ? \]

Well \[ h = x_0 + v_0 t - \frac{1}{2} g t^2 \]

\[ t = \sqrt{2h/g} \]

(\( \text{Can you think of another way to derive this?} \))

\[ \Rightarrow v = - \sqrt{2gh} \]
After collision with the floor (i.e. the earth), what is $\vec{v}$ of the basket ball?

$$\vec{v}_f = -\vec{v}_i = +\sqrt{2gh} \approx \sqrt{20 \cdot \sqrt{h}} \approx 4.5\sqrt{h}$$

To be explicit,

$$V_f = \left(\frac{m_B - m_E}{m_B + m_E}\right) \frac{2m_E}{m_B + m_E} V_i$$

To make it obvious, $x \frac{1}{m_E}$

$$V_f = \left(\frac{m_B}{m_E} - 1\right) \frac{1}{m_E} V_i$$

$$m_B = 0.1 \text{ kg} = 10^{-24} \text{ kg}$$

$$m_E = 10^{-24} \text{ kg}$$

(We needn't keep that many significant digits!)

What about the tennis ball?

$$m_T \downarrow \quad m_B \uparrow$$

Not bucking, so they actually collide after basket ball bounce.

$$V_f = \left(\frac{m_T - m_B}{m_T + m_B}\right)(-v) + \frac{2m_B}{m_T + m_B} (+v)$$

$m_T \sim 0.01 \text{ kg}, \ m_B \sim 0.1 \text{ kg}$

$$V_f = \frac{0.01 - 0.1}{0.11} (-v) + \frac{0.2}{0.11} (+v)$$

$$\approx 0.9v + 2v \approx 3v$$

Tennis ball has $\sim 3$ times the recoil velocity.
How high does it go?

$$\frac{1}{2}mv_f^2 = mgh$$

$$H = \frac{v_f^2}{2g} = \frac{(3v)^2}{2g} = \frac{9(2gh)}{2g} = 9 \text{ times as high as initial}.$$ 

Let me test this with \( h \approx 0.5 \text{ feet} \)

How can this conserve energy?

$$m_Bgh \gg m_Tgh$$

$$\frac{1}{2}m_Bv_f^2 \gg \frac{1}{2}m_Tv_f^2$$

Small change in velocity of basketball = big change in \( v \) of tennis ball.

\[ \overbrace{\begin{array}{c} + \ O \ m_B \\ m_E \text{ at rest} \end{array}} \quad \begin{array}{c} - \ O \ m_T \\ \text{equal velocity.} \end{array} \]

We didn't have to know anything about force other than

- conservative — to allow assumption of conserved KE
- impulsive — jargon for all AP in small \( \Delta t \), to only consider colliding objects not external forces.

Questions on collisions?
Demo 24.10
Moderating neutrons in a fission reactor.

Fission of $^{235}$U produces neutrons moving with a speed of about $10^7$ m/s. (Typically two per fissioning nucleus).

Other $^{235}$U nuclei can be made to fission if they are hit by a neutron with a speed of about $10^3$ m/s.

So to keep a reactor going, you need to slow down the neutrons before they escape.

This is done with a "moderator," a material with which the neutrons collide and lose energy. Suppose it is a nucleus with mass $M_m$. Then the energy of the neutron goes from

$$K_{\text{initial}} = \frac{1}{2} m_n V_{n_i}^2$$

to

$$K_{\text{final}} = \frac{1}{2} M_m V_{n_f}^2 = \frac{1}{2} m_m \left( \frac{m_n - m_m}{m_n + m_m} \right)^2 V_{n_i}^2$$

If head-on ($1$-dim).

The ratio $K_{\text{final}} / K_{\text{initial}} = \left( \frac{m_n - m_m}{m_n + m_m} \right)^2$

To lose energy fast... before escaping... we want most of the energy to be lost, so $m_m \approx m_n$.

Water is best for this... protons in the Hydrogen slow the neutrons very well.

Duterons better because of $p + n \rightarrow ^2H + \gamma$ absorbing neutrons.