Review:

Newton's 3 laws. I'll restate them differently.
Make sure that you understand what they mean.

1). No acceleration without a net force.
   So, $F_{net} = 0$ gives $\ddot{a} = 0 \Rightarrow$ constant velocity,
   and that includes zero velocity.

2). If there is a net force it causes an
   acceleration of $\ddot{a} = \frac{1}{m} F_{net}$.
   Since $F_{net}$ is the sum of all forces,
   $$\ddot{a} = \frac{1}{m} \sum_{i} F_i$$

3). Every force has a reaction force that is
   equal and opposite. The "reaction pair"
   acts on different bodies.
   A pushes on B and B pushes on A.

Now, we'll learn how to use Newton's laws.
In principle it is easy; just figure out what forces
act on what objects, add them and do some algebra
 to find $\ddot{a}$. But the details can be, well, detailed.
So, we need to do a lot of examples. I'll do some.
You do some. Then do more.
First, a very simple example, a ball hanging on a string as shown below. What is the acceleration at this instant?

An immovable ceiling.

First, figure out what forces act on the ball. Do this with a free-body diagram, focusing on one object at a time.

The ball

\[ \text{\textbf{F}_{\text{rope}}} \]

\[ \downarrow \text{mg} \]

Don't complicate it with components just yet. Just list them and indicate directions.

The rope

\[ \text{\textbf{F}_{\text{ceiling}}} \]

\[ \text{mg} \]

For the rope, we have 2 reaction forces. This could get ugly if we have to figure all these out and include the rope's weight. But if the rope's mass is very small---and we approximate it as zero, it becomes simpler. So, we'll use the "massless rope" approximation.

Then we can ignore the weight, and

\[ \text{\textbf{F}_{\text{ceiling}} + \text{F}_{\text{ball}}} = m_{\text{ball}} \alpha = 0. \]

The force is the same on both ends of the rope, it goes nowhere---along its length. It just tightens. This is simplest if we just call it a tension force.
So, \( F_{\text{rope}} = T \) the tension in the rope.

(If the rope has mass, then the tension increases going up along the rope, but we'll be simple for now).

So, the ball's forces are:

\[
\begin{align*}
\text{\( T \)} & \\
\text{\( \downarrow \text{mg} \)}
\end{align*}
\]

Now, we can separate into components. Picking the easiest coordinate system is, well, easiest. We might as well orient one of the axes along one of the forces.

We could do this \( T \) or this \( T \).

It turns out that the second one is easier because we know that there is no acceleration along the string's length. That makes that past trivial.

Now replace \( \text{mg} \) by its components:

\[
\begin{align*}
\text{\( \text{mg} \)} & \\
\text{\( \text{mg} \cos \theta \)} & \\
\text{\( \text{mg} \sin \theta \)}
\end{align*}
\]
So, use Newton's second law for each dimension.

\[ \vec{a} = \frac{1}{m} \sum \vec{F} \Rightarrow \begin{cases} a_y = \frac{1}{m} \sum F_y \\, \, \, \Rightarrow \, \, \, a_y = \frac{1}{m} \sum F_y \, \, \, \Rightarrow \, \, \, a_y = \frac{1}{m} \left( T - mg \cos \theta \right) = 0 \end{cases} \]

\[ \Rightarrow T = mg \cos \theta \]

And

\[ a_x = \frac{1}{m} \left( -mg \sin \theta \right) \]

\[ \therefore a_x = -g \sin \theta \]

Check: Units, \( g \to 0 \), \( \theta \to 0 \), \( \theta \to \pi/2 \).
A more involved, but still simple, example.
Two masses hang from a pulley.

Proceed as before:
1. Free body diagram for
2. each object separately.
3. Pick a coordinate system.
4. Take advantage of constraints.
5. Algebra.
6. Validation.

1). Free body diagrams.

\[ T_1 \]
\[ m_1 \]
\[ m_1 g \]

\[ T_2 \]
\[ T_3 \]
\[ m_2 \]

\[ m_2 g \]

where we have assumed that the rope is massless in order to get the same tension at the bottom and top of each section of rope.

If the pulley is light -- approximately zero mass, then \( T_1 = T_2 \). This comes about due to something relying on torque that we'll see later. For now, just remember that

\[ T_1 = T_2 \text{ around a pulley iff it is massless.} \]

(Note: iff is an abbreviation for "if, and only if").
So, \( T_3 = T_1 + T_2 \) since the pulley doesn't move, i.e., it doesn't accelerate.

Let's simplify the notation with

\[
T = T_1 = T_2
\]

Then,

\[
\begin{array}{c}
\uparrow T \\
\downarrow 1 \\
\downarrow m_1 g
\end{array}
\quad \begin{array}{c}
\uparrow T \\
\downarrow 2 \\
\downarrow m_2 g
\end{array}
\]

2) Pick a coordinate system. Easy & obvious here.

\[
\Rightarrow a_{yy} = \frac{1}{m_1} (T - m_1 g)
\]

\[
\hat{\dot{y}} = \frac{1}{m_2} (T - m_2 g)
\]

This double subscripts gets annoying, so I prefer to use \( \hat{y} \) instead of \( a_y \). So,

\[
\hat{y}_1 = \frac{1}{m_1} (T - m_1 g)
\]

\[
\hat{y}_2 = \frac{1}{m_2} (T - m_2 g)
\]

We have two equations and three unknowns. Hmpt!
So, step 3, apply constraints.

3). If $m_1$ goes up some distance then $m_2$ goes down the same distance, assuming the rope stays taught of course.

$$\Rightarrow \ddot{y}_1 = -\ddot{y}_2 \quad \text{and} \quad \ddot{y}_1 = -\ddot{y}_2$$

This constraint gives us our third equation.

The rest is just algebra.

4). Plug-n-chug.

$$\ddot{y}_1 = -\ddot{y}_2 \Rightarrow \frac{1}{m_1}(T - m_1 g) = -\frac{1}{m_2}(T - m_2 g)$$

$$\Rightarrow \frac{T}{m_1} + \frac{T}{m_2} = \frac{m_2 g + g}{m_2}$$

$$\Rightarrow T \left( \frac{m_1 + m_2}{m_1 m_2} \right) = 2g$$

$$T = \frac{2g m_1 m_2}{m_1 + m_2}$$

Check: units

limits

$g \to 0$
$m_1 \to 0$
$m_2 \to 0$
$m_1 = m_2 = m$
So,

\[ \ddot{y}_1 = \frac{T}{m} - g = \frac{a g m_2}{m_1 + m_2} - g \]

\[ \ddot{y}_1 = g \left( \frac{2m_2 - m_1 - m_2}{m_1 + m_2} \right) \]

\[ \ddot{y}_1 = \frac{m_2 - m_1}{m_1 + m_2} g \]

And

\[ \ddot{y}_2 = \frac{m_1 - m_2}{m_1 + m_2} g \]

5). Validation

- **Units.** ✓
- **Limits.**  \( g \to 0 \Rightarrow \ddot{y} \to 0 \) equals turning off all weights.

\[ m_1 = m_2 \Rightarrow \ddot{y} = 0 \text{ they balance.} \]

\[ m_2 = 0 \Rightarrow \ddot{y}_1 = -g \text{ it just falls.} \]

\[ m_1 = 0 \Rightarrow \ddot{y}_2 = -g \]

\[ m_1 = 0 \Rightarrow \ddot{y}_1 = +g \] (An artifact of our constraint)

And, of course, this is obvious... inertias odd and weight balance.
A more complicated example. Find the acceleration of both blocks in the configuration below.

Assume light (i.e., massless) ropes and pulleys.

You solve this exactly as before:

1). Free-body diagrams for each object separately.
2). Pick a coordinate system (or systems).
3). Take advantage of constraints.
4). Algebra
5). Validation.
1). Free-body diagrams.

We could similarly find the tensions for the other pulley... but not necessary. You should be able to understand why once this is all familiar.

Note that we used the same tension around the pulley -- assuming it is massless.

And, we can see that $T_3 = T_2$ since the rope goes around to $m_2$.

Since the pulley has no mass,

$$(m=0) \quad a = 2T_2 - T_1 \quad \Rightarrow \quad T_1 = 2T_2$$

Relabel the diagrams for simplicity, $T_2 \rightarrow T$.
2). Define a coordinate system.
Simple here. ty is up. Origin doesn't matter since it is only changes that matter in the acceleration.

So, Newton's 2nd law gives

\[ \ddot{y}_1 = \frac{1}{m_1} (a_T - m_1 g) = \frac{a_T}{m_1} - g \]

and

\[ \ddot{y}_2 = \frac{1}{m_2} (T - m_2 g) = \frac{T}{m_2} - g \]

(Beware the temptation to think that you can generalize this to something like "two ropes on mass 1 so 2T instead of 1T". Make sure you understand why, because things will complicate this later).

3). Take advantage of constraints.
Here it is not so obvious how the rope constrains the relative motion of the two objects.

Puzzling about it a bit shows that if \( m_1 \) moves down some distance \( m_2 \) moves up twice as far. I.e. \( \ddot{y}_2 = -2 \ddot{y}_1 \)

But let's see why, rigorously.
The constraint is essentially that the rope doesn't stretch.

That just means that its length doesn't change.

So, if we write an expression of its length, \( L \), then \( \frac{dL}{dt} = 0 \) means it doesn't change.

So, define the positions in our coordinate system and then express \( L \) in terms of them.

\[
L = y_1 - d + \pi R_1 + (y_1 - d - h) + \pi R_2 + y_2 - h
\]

\[
\Rightarrow \frac{dL}{dt} = \dot{y}_1 + 0 + 0 + \dot{y}_1 + 0 + 0 + \dot{y}_2 + 0 = 0 \quad \text{required}
\]

\[
\Rightarrow 2\dot{y}_1 + \dot{y}_2 = 0 \quad \Rightarrow \quad \dot{y}_2 = -2\dot{y}_1
\]

One more derivative gives

\[
\ddot{y}_2 = -2\ddot{y}_1
\]
This is a complicated way to get what we already guessed, but it is a rigorous way to be explicitly correct. It will work no matter how complicated the configuration becomes. And it clarifies what the constraint really means.

4). Algebra. We have 3 equations and three unknowns. Just chug.

But there is some elegance in algebra done simply.

Note that we can rewrite the eqns.

\[ \ddot{y}_1 = \frac{2T}{m_1} - g \quad \Rightarrow \quad T = \frac{1}{2} m_1 (\ddot{y}_1 + g) \]

and \[ \ddot{y}_2 = \frac{T}{m_2} - g \quad \Rightarrow \quad T = m_2 (\ddot{y}_2 + g) \]

and \[ \ddot{y}_2 = -2 \ddot{y}_1 \quad \text{plug in} \]

\[ T = m_2 (g - 2 \ddot{y}_1) \]

so \[ \ddot{y}_1 = \frac{2T}{m_1} - g = \frac{2m_2}{m_1} (g-2 \ddot{y}_1) - g \]

\[ \Rightarrow \quad \ddot{y}_1 = -4 \frac{m_2}{m_1} \ddot{y}_1 + g \left( \frac{2m_2}{m_1} - 1 \right) \]

\[ \ddot{y}_1 \left( 1 + 4 \frac{m_2}{m_1} \right) = g \left( \frac{2m_2}{m_1} - 1 \right) \]
\[ y' = \frac{2m_2}{m_1} \frac{1}{1 + \frac{4m_2}{m_1}} \]

\[ \ddot{y}_1 = g \left( \frac{2m_2 - m_1}{m_1 + 4m_2} \right) \]

and

\[ \ddot{y}_2 = -2g \left( \frac{m_1 - 2m_2}{m_1 + 4m_2} \right) \]

5). Validation.

Units. \( \checkmark \)

Limits. \( g > 0 \)

\( m_1 = 2m_2 \Rightarrow \) no acceleration.

Why? The other force comes from the ceiling pulling on the other end of the rope.

Cool. Multiple loops make it easier to lift. = "Block & tackle."
Example 4

Intentionally challenging to demonstrate various points, especially how to take it one simple step at a time.

No friction.

What will happen?

Slide in direction of larger effective weight.

How to calculate \( \ddot{a} \)?

Determining forces.

\[ a = \frac{\theta}{E/m} \]


Constraint is \( x_1 + x_2 = l \)

\[ \frac{dl}{dt} = \dot{x}_1 + \dot{x}_2 = 0 \]

\[ \ddot{x}_1 = -\ddot{x}_2 \]

\[ \frac{1}{2} \dddot{x}_1 = -\dddot{x}_2 \]

Forces

\[ F \]

\[ T_1 \]

\[ W_{\text{eff}} \]

\[ m_g \cos \theta_1, \]

\[ m_g \sin \theta_1, \]

\[ m_g \]

\[ W_{\text{eff}} = m_g \sin \theta_1, \]

What is \( \dddot{x}_1 \)?

\[ m \dddot{x}_1 = W_{\text{eff}} - T_1 = m_g \sin \theta_1 - T_1 \]

\[ m \dddot{x}_1 = m_g \sin \theta_1 - T_1 \]
Now, you do the same for $m_2$.

\[ m_2 \ddot{x}_2 = m_2 g \sin \theta_2 - T_2 \]

Note the signs. It depends on our choice of coordinate system.

**$T_1 = T_2 = T$ if and only if, rope is massless.**

So, solve for $\dot{x}_1$ and $\ddot{x}_2$. First, solve for $\ddot{x}_1$ plug in $T$.

\[ T = m_2 (g \sin \theta_2 - \ddot{x}_2) \quad \text{Units ok?} \]

\[ m_1 \ddot{x}_1 = m_1 g \sin \theta_1 - m_2 g \sin \theta_2 + m_2 \ddot{x}_2 \]

\[ (m_1 + m_2) \ddot{x}_1 = (m_1 \sin \theta_1 - m_2 \sin \theta_2) g \]

\[ \ddot{x}_1 = \frac{(m_1 \sin \theta_1 - m_2 \sin \theta_2) g}{m_1 + m_2} \]

Does this make sense? If $\theta_1 = \theta_2 = 0$, $\ddot{x}_1 = 0$

If $\theta_1 = \theta_2 \neq 0 \frac{1}{2}$ $m_1 = m_2$, $\ddot{x}_1 = 0$

If $m_1 = 0$, $\theta_2 = 90^\circ$, $\ddot{x}_1 = g$

If $\theta_1 = 0 \neq \theta_2 = 90^\circ$, $\ddot{x}_1 = -m_2 g$

You can solve for $\dot{x}_2 \neq T$.
Example 6

Intuitively (or math'ly)

You figure out constraints:

\[ L_1 = P_1 + \pi R_1 + P_2 - x_1 \Rightarrow \dot{L}_1 = 0 = 2\dot{P}_1 + 0 - \dot{x}_1 \Rightarrow \ddot{x}_1 = 2\dot{P}_1 \]

\[ \dot{x}_2 = \frac{1}{2} \dot{x}_1 \]

Now, what are the forces?

\[ m_2 \ddot{x}_2 = m_2 g - T_2 \quad \text{and} \quad m_1 \ddot{x}_1 = T_1 \]

But \( T_2 \) is shared across 2 halves of rope 1. So, \( T_1 = \frac{1}{2} T_2 \)

\[ m_1 \ddot{x}_1 = \frac{1}{2} T_2 = \frac{1}{2} \left( m_2 g - m_2 \ddot{x}_2 \right) = \frac{1}{2} m_2 g - \frac{1}{2} m_2 \cdot \frac{1}{2} \ddot{x}_1 \]

\[ \left( m_1 + \frac{m_2}{4} \right) \ddot{x}_1 = \frac{1}{2} m_2 g \quad \Rightarrow \quad \ddot{x}_1 = \frac{1}{2} \frac{m_2 g}{m_1 + \frac{m_2}{4}} \]

T's & x's are similar
Conceptual questions

1). When you are standing in an elevator that is accelerating upward, you feel heavier. And, in fact if you were standing on a scale, it would measure a force larger than when the elevator is not moving. Does your weight increase? No. Then why does the scale read a larger force? mg + ma.

2). If a car gets hit by a [train], what is the relationship between the force that each experiences?

3). Is it possible to have motion in the absence of force?

4). Is it possible to have force without motion?

5). In a train, each car is connected by a coupler. Does the tension in those couplers increase, decrease, or stay the same as you move from the locomotive to the caboose? Decreases. (If accelerating...) only need to pull what is behind you.