Last time we talked about motion in 1D.

\[ x(t) = \text{position at time } t \]
\[ v(t) = x'(t) \]
\[ a(t) = x''(t) \]

Or, go other way with integrals.

We did several example problems using constant acceleration e.g. with free-falling objects.

The key is to:

- Set up a coordinate system, position \& time
- Find \( x(t) \), e.g. from \( a(t) \). If constant, \( a(t) = A \) then just familiar \( x(t) = x_0 + v_0 t + \frac{1}{2} At^2 \)
  but we need to make sure we get \( x_0, v_0, \) and \( A \) correct for our coordinate system.
- Solve for what we want in terms of time or velocity, etc.
For example, suppose we throw a ball up vertically with an initial speed of 10 m/s. How high does it go?

Coord system: \( y=0 \) at release, \( t=0 \) at release, \( \hat{y} \) pointing up.

Eqn. of motion: 
\[
y(t) = y_0 + v_0 t - \frac{1}{2} g t^2
\]

with \( v_0 = +10 \text{ m/s} \), \( \hat{y} \) because \( \hat{y} \) is up.

Maximum height = ? Actually the easiest way to find that is with velocity. The max height is reached when the ball stops going up and heads back down \( \Rightarrow v=0 \).

\[
V(t) = 0 \text{ at what } t?
\]

\[
V(t) = \dot{y}(t) = v_0 - gt \quad \text{so} \quad t_{\text{top}} = \text{time at top.}
\]

\[
\Rightarrow 0 = v_0 - gt_{\text{top}} \quad \Rightarrow \quad t_{\text{top}} = \frac{v_0}{g}
\]

And \( y(t_{\text{top}}) = v_0 t_{\text{top}} - \frac{1}{2} g t_{\text{top}}^2 = \frac{v_0^2}{g} - \frac{1}{2} g \frac{v_0^2}{g^2} = \frac{v_0^2}{2g} \)
Now, let's generalize this to >1 dimension. Velocity is time derivative of position, also known as rate of position change.

In 3D, position is not just \( x \), but also \( y \) and \( z \) coordinates.

That is, position is a vector.

For example, the position of a ball in some coordinate system is

\[
\mathbf{b}(t) = (b_x, b_y, b_z) = b_x \hat{x} + b_y \hat{y} + b_z \hat{z}
\]

Or, we could write it as \( \mathbf{b} = b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k} \).

By definition, velocity is \( \frac{d\mathbf{b}}{dt} \)

\[
\mathbf{v} = \frac{d}{dt} \left( b_x \hat{x} + b_y \hat{y} + b_z \hat{z} \right)
\]

\[
= \frac{db_x}{dt} \hat{x} + \frac{db_y}{dt} \hat{y} + \frac{db_z}{dt} \hat{z}
\]

\[
= v_x \hat{x} + v_y \hat{y} + v_z \hat{z}
\]

Similarly, \( \mathbf{a} = \frac{d\mathbf{v}}{dt} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \)
So, in 3D we just get three different independent equations, one for each dimension.

\[ \vec{b}(t) = b_x(t) \hat{x} + b_y(t) \hat{y} + b_z(t) \hat{z} \]

\[ \vec{v}(t) = b_x(t) \hat{x} + b_y(t) \hat{y} + b_z(t) \hat{z} \]

\[ \vec{a}(t) = b_x(t) \hat{x} + b_y(t) \hat{y} + b_z(t) \hat{z} \]

\[ \stackrel{\uparrow}{\hat{x}}: a_x, v_x \text{ depend only on } b_x. \]

\[ \stackrel{\uparrow}{\hat{y}}: a_y, v_y \text{ depend only on } b_y. \]

\[ a_z, v_z \text{ depend only on } b_z. \]

The same is true for getting \( \vec{b}(t) \) from \( \vec{a}(t) \).

If I drop a ball, it accelerates downward, so \( a_z(t) = -g \) (where this assumes \( +\hat{z} \) is up).

What if I throw a ball horizontally, e.g. by letting it roll off of a table?
Call \( t=0 \) the instant it leaves the edge with a coordinate system as below:

What is \( \vec{b}(t) \)?

The \( z \) component is familiar old free-fall,

\[
a_z = -g.
\]

Actually, that is the only acceleration (assuming no air resistance). So,

\[
\vec{a}(t) = -g \hat{z} = 0 \hat{x} + 0 \hat{y} - g \hat{z}
\]

\[
\vec{v}(t) = \int \vec{a}(t) \, dt = C_1 \hat{x} + C_2 \hat{y} + (C_3 - g t) \hat{z}
\]

\[
= v_{x_0} \hat{x} + v_{y_0} \hat{y} + (v_{z_0} - gt) \hat{z}
\]

\[
\vec{b}(t) = \int \vec{v}(t) \, dt = (v_{x_0} t + C_4) \hat{x} + (v_{y_0} t + C_5) \hat{y} +
\]

\[
+ (C_6 + v_{z_0} t - \frac{1}{2} gt^2) \hat{z}
\]
\[ \vec{b}(t) = (x_0 + v_{x_0} t) \hat{x} + (y_0 + v_{y_0} t) \hat{y} + (z_0 + v_{z_0} t - \frac{1}{2} g t^2) \hat{z} \]

\[ = x(t) \hat{x} + y(t) \hat{y} + z(t) \hat{z} \]

3 independent equations of motion

What if it has a speed of about 2 m/s as it leaves the edge, what is $\vec{b}(t)$ then?

\[ \vec{b}(t) = 0 \hat{x} + (2 \hat{y}) \hat{y} + (h - \frac{1}{2} g t^2) \hat{z} \]

Where does it hit the floor?

Well to answer that, we can first find when it hits the floor, call it $t = t_f$. Then where is $\vec{b}(t_f)$.

\[ b_z(t_f) = 0 \Rightarrow h - \frac{1}{2} g t_f^2 = 0 \]

\[ \Rightarrow t_f = \sqrt{\frac{2h}{g}} \]

\[ \Rightarrow b_y(t_f) = 2 \frac{m}{s} \sqrt{\frac{2h}{g}} \]

If $h \sim 1 \text{m}$, $b_y(t_f) \sim 2 \sqrt{\frac{2}{\sqrt{10}}} \sim \frac{2.8}{3} \sim 1 \text{m}$.
Note that the time it takes to reach the floor doesn't depend on how fast it is moving in the y-direction. That is because the 3 directions have independent equations of motion.

We can demonstrate this with a demo:

This setup drops the green ball (by pulling out a support rod placed thru a hole in its center). Simultaneously, it kicks the orange ball off its platform. So the orange ball has an initial velocity in the horizontal direction but the green ball does not.

Nonetheless, you can hear that they hit the ground simultaneously. That happens because only the acceleration due to g affects the vertical fall time.
What if we launched a ball upward with some angle, instead of horizontally?

That would give it velocity components both horizontally and vertically.

If the initial speed (i.e. magnitude of velocity) where \( |\vec{V}_0| = V_0 \), then,

\[
V_{x0} = V_0 \sin \theta \\
V_{y0} = V_0 \cos \theta \\
V_{x0} = 0
\]

Note that \( \theta = 90^\circ \) reduces to our old fashioned 1D motion.

What if it was launched at some other angle, \( \phi \), between the x and y-axis?

Then, it would be best to just redefine the coordinate system with \( y \), or \( x \), along the initial direction. That makes it 2D instead of 3D. Choose simplest coordinate system!
It is worth noting that the velocity along the y direction, $v_y$, is just

$$v_{yo} = \vec{v}_o \cdot \hat{y} = |\vec{v}_o| |\hat{y}| \cos \theta = |\vec{v}_o| \cos \theta$$

and

$$v_{zo} = \vec{v}_o \cdot \hat{z} = |\vec{v}_o| |\hat{z}| \cos(90^\circ - \theta) = |\vec{v}_o| \sin \theta$$

and

$$v_{xo} = \vec{v}_o \cdot \hat{x} = |\vec{v}_o| |\hat{x}| \cos(90^\circ) = 0.$$  

This is, in fact always true as $\vec{v}(t)$ changes... the usefulness of the dot product.

As the ball falls, $v_y$ will stay the same, but $v_z$ changes due to $g$.

$$v_{y(t)} = v_{yo} \quad \text{and} \quad v_{z(t)} = v_{zo} - gt$$

So, $$|\vec{v}(t)| = \sqrt{v_{y(t)}^2 + v_{z(t)}^2} \quad \text{increases with time.}$$

$$|\vec{v}(t)| \neq 0 \quad \text{at top anymore}$$

Only $v_z = 0$ at the top. $v_y$ is still just $v_{yo}$. 
Suppose I launch a ball from the ground giving it an initial velocity \( \vec{v}_0 \) upward at some angle \( \theta \) from the horizontal. Where does it hit the ground again? (Assume that the ground is level.)

\[
\vec{v}_0 = v_0 \cos \theta \hat{x} + v_0 \sin \theta \hat{y}
\]

where \( v_0 = \abs{\vec{v}_0} \)

Note that I've changed from having the motion in the \( yz \) plane to now being in the \( xy \) plane. Make sure you feel comfortable with that.

\[
\vec{v}(t) = v_0 \cos \theta \hat{x} + (v_0 \sin \theta - gt) \hat{y}
\]

\[
\vec{b}(t) = \left[ x_0 + (v_0 \cos \theta) t \right] \hat{x} + \left[ y_0 + (v_0 \sin \theta) t - \frac{1}{2} gt^2 \right] \hat{y}
\]

\[
= x(t) \hat{x} + y(t) \hat{y}
\]

old fashioned 1D free-fall

old fashioned constant velocity 1D motion.

To find where it hits the ground, first find when it hits the ground.
\[ y(t_g) = 0 \Rightarrow y_0 + (v_0 \sin \theta) t_g - \frac{1}{2} g t_g^2 = 0 \]

\[ \Rightarrow t_g \left( v_0 \sin \theta - \frac{g t_g}{2} \right) = 0 \]

\[ \Rightarrow t_g = 0 \quad \text{(obviously at the start)} \]

or

\[ t_g = \frac{2v_0 \sin \theta}{g} \]

Position in \( x \) at that time is

\[ x(t_g) = x_0 + (v_0 \cos \theta) t_g = \frac{2v_0^2 \cos \theta \sin \theta}{g} \]

\[ = \frac{v_0^2}{g} \left[ 2 \sin \theta \cos \theta \right] \]

\[ = \frac{v_0^2}{g} \sin 2\theta \quad \text{by trig identity.} \]

Check units, and limits.

Draw motion and trajectory.

\[ x(t), \dot{x}(t), \ddot{x}(t) \]

\[ y(t), \dot{y}(t), \ddot{y}(t) \]

\[ y(x) \]
What angle gives the maximum range, for a fixed $v_0$?

There are two ways to do this.

First, just plot $\sin 2\theta$ and find $\theta$ that gives max.

\[
\sin 2\theta \rightarrow \theta \rightarrow \text{Max range if } \theta = 45^\circ
\]

\[
\frac{\pi}{4} \quad \frac{\pi}{2} \quad \pi
\]

\[
45^\circ \quad 90^\circ \quad 180^\circ
\]

Second, we can find max with $\frac{dx}{d\theta} = 0$.

\[
\frac{d}{d\theta} \left[ \frac{v_0^2}{g} \sin 2\theta \right] = \frac{v_0^2}{g} 2 \cos 2\theta = 0
\]

$\Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \pi/2 \Rightarrow \theta = \pi/4 = 45^\circ$

And, of course, the graph above shows that $\sin 2\theta$ has 0 slope at $\theta = 45^\circ$ as it must.
What if I throw a ball straight up while running?

From my point of view, it goes straight up and back down.

But from your point of view, I have a horizontal velocity component, so the trajectory is

\[ x(t) = x_0 + v_{x_0} t \]
\[ v_x(t) = v_{x_0} \]
\[ y(t) = y_0 + v_{y_0} t - \frac{1}{2} g t^2 \]
\[ v_y(t) = v_{y_0} - gt \]

And, you see that I have the same \( x(t) \) as the ball.

The relative velocity of the ball, relative to my coordinate system, is just

\[ \vec{b}_m(t) = \vec{b}_i(t) - \vec{M}_i(t) \]

\( \rightarrow \) Me in your coordinate system
\( \rightarrow \) Ball in your coordinate system
\( \rightarrow \) Ball in my coordinate system.

This is just like the 1D relative motion from last week, but with vector signs.
Demo: Ball over funnel.

Demo: Shoot a falling target, with simultaneous release.

ball → \( \vec{b}(t) = [x_0 + v_0 \cos \theta \cdot t] \hat{x} + 
                        [y_0 + v_0 \sin \theta \cdot t - \frac{1}{2} g t^2] \hat{y} \)

target → \( \vec{T}(t) = D \hat{x} + [(H - \frac{1}{2} g t^2) \hat{y} \)

To hit it, I want \( \vec{b}(t) = \vec{T}(t) \) at some time \( t \), before the target reaches the ground.

\( \vec{b}(t) = \vec{T}(t) \) means:

\[ v_0 \cos \theta \cdot t = D \]

\[ v_0 \sin \theta \cdot t - \frac{1}{2} g t^2 = H - \frac{1}{2} g t^2 \]

The acceleration due to gravity cancels!

Divide \( \Rightarrow \frac{v_0 \cos \theta \cdot t}{v_0 \sin \theta \cdot t} = \frac{D}{H} \)

\[ \tan \theta = \frac{H}{D} \]

\( \Rightarrow \) Aim directly at it! Regardless of \( v_0 \)!

Of course, if \( v_0 \) too small, they hit below the floor.