 Describe motion with \( x(t) \).

\[
V(t) = \frac{dx(t)}{dt} = \dot{x}(t)
\]

\[
a(t) = \frac{dV(t)}{dt} = \ddot{v}(t) = \dddot{x}(t)
\]

So, given \( a(t) \) we can derive \( x(t) \) by integrating.

A simple case is \( a(t) = A \), a constant.

\[
V(t) = \int a(t) \, dt = At + V_0
\]

\[
x(t) = \int V(t) \, dt = \frac{1}{2}At^2 + V_0t + x_0
\]
An example problem:

You walk into a room at 1 m/s and see a roach 1 m in front of you. It scurries away toward a fridge that is 2 m from it. The roach moves at 2 m/s.

If you had constant acceleration, what acceleration do you need to squash the roach before it reaches safety under the fridge?

1). Coordinate system. $t=0$ when you see the roach. $x=0$ is your position at $t=0$

2). Equation of motion:

Roach $x(t) = x_0 + v_0 t + \frac{1}{2} a_r t^2$ so $a_r = 0$.

You $x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$

$= v_0 t + \frac{1}{2} a t^2$

We know $l$, $v_r$, $v_0$ but let's not plug in yet.
When do you reach the roach? When \( x(t) = r(t) \).

\[
v_0 t + \frac{1}{2} a t^2 = l + v_r t
\]

\[
\Rightarrow A = \frac{2}{t^2} \left[ l + v_r t - v_0 t \right]
\]

Check units

Bigger \( t \) \( \Rightarrow \) Bigger \( A \).

EtC.

So, if I know \( t \), I can get \( A \).

What \( t \)? Small enough to be before it reaches the fridge. When is that?

When \( s(t) = l \).

\[
L = l + v_r t_s \quad t_s = t_{\text{safety}}
\]

So,

\[
t_s = \frac{(L - l)}{v_r} \quad \text{Units OK? Sensible?}
\]

\[
A = 2 \left[ \frac{l + v_r (L-l) - \frac{v_0 (L-l)}{v_r}}{v_r} \right]
\]

\[
= \frac{2 v_r^2}{(L-l)^2} \left[ L - \frac{v_0 (L-l)}{v_r} \right]
\]

Units?

Limits?

\( L = l \)

\( v_r = 0 \)

\( v_0 = 0 \)?

\[
= 4 \text{ m/s}^2
\]
How fast would you be going when you reached the reach?

Constant acceleration, so

\[ v(t) = v_0 + at \]

\[ = v_0 + 4 \text{ m/s}^2 \cdot \left( \frac{L-x}{v_c} \right) \]

\[ = 1 \text{ m/s} + 4 \text{ m/s} = 5 \text{ m/s} = 10 \text{ MPH} \]

a full run, OUCH.

\[ \text{FREE-FALL} \]

Last time, we derived \( a(t) \) for a mass on a spring by observing \( y(t) \). Let's do the same for free-fall.

Observe

\[ y(t) = \text{parabola}, \]
\[ = y_0 - ct^2 \]

\[ \Rightarrow v(t) = -2ct \]
\[ a(t) = -2c \]

\[ \Rightarrow \text{Acceleration is constant in free-fall.} \]

Let's call that constant \( \text{accel} = -g \) (i.e. \( 2c = g \))
So, $a(t) = -g$

$V(t) = -gt + V_0$

$y(t) = -\frac{1}{2}gt^2 + V_0t + y_0$

I prefer $y(t) = y_0 + v_0t - \frac{1}{2}gt^2$

$g$ is measured to be $9.80 \text{ m/s}^2 \approx 32 \text{ ft/s}^2 \approx 25 \text{ MPH}^2$.

So we now know how to describe free-fall, even if we don't know why $a$ is constant. But, let's jump ahead and see why.

$F = \frac{GMm}{r^2} = ma$

so, $a = \frac{GM}{r^2} \rightarrow$ doesn't depend on $m$.

$r = 6 \times 10^6 \text{ m}$ so $r \pm 1 \text{ m}$ is a constant.

Don't need to know that "why" to predict freefall on earth, but important to extrapolate to mars.
Air resistance

Air resistance changes this a bit. For some objects it changes a lot. E.g. feather v.s. ball.

But, we can remove the air and see that they fall at the same rate.

Now, graph full $x(t)$, $v(t)$, $a(t)$ of bouncing ball.

![Graphs showing position, velocity, and acceleration over time.](image)
Sample Problem.

If you drop a stone off the golden gate bridge tower, how long does it take to pass the roadway? How long to hit the water?

Well, via google:

750ft $\sim 230\text{ m}$  \hspace{1cm} $x_0 = 230\text{ m}$

250ft $\sim 80\text{ m}$  \hspace{1cm} $R = 80\text{ m}$

0  \hspace{1cm} $W = 0\text{ m}$

And:

$A = g = -10\text{ m/s}^2$

What is the equation of motion?

$x(t) = x_0 + v_0 t + \frac{1}{2} A t^2$

$= x_0 + 0 - \frac{1}{2} g t^2$

Water at:

$0 = x_0 - \frac{1}{2} g t^2$

$t = \sqrt{\frac{2x_0}{g}} \approx \sqrt{\frac{460}{10}} \approx 7\text{ seconds}$

Check units check limits

What if I had "g"?

6.85 to be more precise,

or 6.9 to be correctly precise.
Roadway?

\[ 80 = 230 - \frac{1}{2} \cdot g \cdot t^2 \]

\[ \frac{1}{2} \cdot g \cdot t^2 = 230 - 80 = 150 \]

\[ t^2 = \sqrt{\frac{300}{g}} \approx \sqrt{30} \approx 5.5 \]

\[ = 5.53 \approx 5.5 \] to correct precision.

Note that it would have been easier to just plug in \( x = 0 \) in the previous step:

\[ t = \sqrt{2(x_0 - x)} / g \]

Velocity?

\[ v(t) = v_0 + at = -gt \]

\[ \Rightarrow \approx -54 \text{ m/s at roadway} \approx 110 \text{ MPH} \]

\[ \approx -67 \text{ m/s at water} \approx 140 \text{ MPH} \]

What do the minus signs mean?

What if I changed the coordinates so

\[ x_0 = 0 \quad x = 230 \text{ m} \quad R = 230 - 80 = 150 \text{ m} \]

That actually makes it easier, because \( x_0 \) can always be dropped just like \( v_0 \).

\[ \Rightarrow x(t) = \frac{1}{2} \cdot gt^2 \Rightarrow t = \sqrt{\frac{2x(t)}{g}} \] Simpler.
Can measure height with free-fall.

\[ y(t) = h + v_0 t - \frac{1}{2} g t^2 \]

\[ 0 = h - \frac{1}{2} g t^2 \]

\[ h = \frac{1}{2} g t^2 \]

If you counted 10 seconds,

\[ h = 490 \text{ m} \quad \text{(Quick estimate = 500)} \]

(Beware that speed of sound is 340 m/s so if we listen rather than look, we'll be off because \( h = \frac{1}{2} g (t - h/340)^2 \) )

Velocity at ground = \(-gt = -98 \text{ m/s} \approx -200 \text{ MPH} \)

⇒ Please don't drop rocks off sky-scrapers!
What if you were on the ground looking up? How to measure height?

Well could throw the stone up.

\[ x(t) = x_0 + V_0 t - \frac{1}{2} g t^2 \]

Find \( V_0 \) that makes the stone just hit the top, or time it.

We'd find \( V_0 = +98 \text{ m/s} \) and that it takes 10 s to reach the top... and another 10 to fall back down.

Have students graph \( x, v, a \) v.s. \( t \).

Note: Same motion up as down.
Reaction Time

Suppose I drop a $1 bill.

You can probably see the effect of air resistance, but at first, it is simple free-fall.

It starts slow, \( V_0 = 0 \), so if I'm quick, I can catch it.

\[ \text{half } L \sim 4 \text{ thumbs } \sim 4'' \sim 10 \text{ cm } = 0.01 \text{ m} \]

\( t \) to fall that far?

\[ 0.01 \text{ m} = \frac{1}{2} g t^2 \]

\[ t = \sqrt{\frac{0.02}{g}} \approx 0.05 \text{ seconds.} \]

Pretty quick.

I need a volunteer to test your reflexes. You catch it, you keep it.
Measure reaction time with a ruler.

\[ x(t) = -\frac{1}{2}gt^2 + x_0 \]

\[ x(t) - x_0 = -\frac{1}{2}gt^2 \]

\[ t = \sqrt{\frac{2(x_0 - x(t))}{g}} \]

Measure \( x(t) \) gives us \( t \) ... can measure it even if it is a very small time.

Comment on precision of measurement.

... could repeat it many times.
Sky-Divers

Suppose two sky divers jump out of a hovering helicopter 1 second apart. What is the equation of motion of each? Do they stay the same distance apart?

\[
x_1(t) = h + 0t - \frac{1}{2}gt^2
\]
\[
x_2(t) = h - \frac{1}{2}g(t-1)^2 \quad \text{for } t \geq 1.
\]

\[
\Delta x = x_2(t) - x_1(t)
\]
\[
= h - \frac{1}{2}g(t-1)^2 - h + \frac{1}{2}gt^2
\]
\[
= \frac{1}{2}g \left( t^2 - (t-1)^2 \right)
\]
\[
= \frac{1}{2}g \left[ t^2 - t^2 + 2t - 1 \right]
\]
\[
= \frac{1}{2}g \left[ 2t - 1 \right] = gt - \frac{1}{2}g
\]

No, their separation increases.
An easy way to see this is to think about their velocity.

\[ v_1(t) = v_0 t - gt \]

\[ v_2(t) = -g(t-1) \quad \text{for } t \geq 1 \]

\[ = -gt + g \cdot 1 \]

\[ v_1(t) \] is a constant 10 m/s faster (negatively).

Their velocities rise (decrease) in step with each other, but \( v_1 \)'s "head start" in velocity makes him continually move away from \#2.
Relative Motion

We encounter situations like that often, where objects are moving relative to each other.

Rather than fixing the coordinate system on the earth, we can put it on one of the moving objects.

E.g. One sky diving from a rising helicopter.

From the ground, the helicopter has $v = +2 \text{ m/s}$. The sky diver has the same while he's inside it. But what happens when he steps out. He falls with $a(t) = -g$.

$$x(t) = x_o + v_o t - \frac{1}{2} g t^2$$

Graph $x$ vs $t$

It takes a while before $-g$ overtakes the $+2 \text{ m/s}$ of velocity. (Could calculate it)

How?
If our coord sys was on the helicopter, then, we would not see the guy go up then down, of course. He just falls.

\[ x(t) = -\frac{1}{2}gt^2 \]

If our coord system were on the diver, he sees the helicopter accelerate up, away from him

\[ x_h(t) = +\frac{1}{2}gt^2 \quad v_0 = 0 \quad a = \text{positional} \]

and the ground moving toward him at

\[ x_g(t) = v_0 t - \frac{1}{2}gt^2 \]

These all describe the same motion, just from different viewpoints.

We call these reference frames... a coordinate system and a clock.

In "normal", everyday situations this is simple. Relativity will complicate it... but that is much later.
Consider again the two skydivers. The second guy sees the first moving away with a constant $v = +g \cdot t$ because of his 1 second head start.

This is an accelerated reference frame. He has the same acceleration as the other guy so they don't accelerate relatively.

**Vomit Comet**

Free-falling airplane means passengers can free-fall & stay inside.

\[ a_{\text{Pass}} = a_{\text{Plane}} \]

so no relative acceleration $\Rightarrow$ feel weightless.
Position of 2 in 1's RF is +L

Position of 1 in 2's RF is -L

If velocity of 2 in 1's RF is +V
Then velocity of 1 in 2's RF is -V

Because V is just dL/dt.

It doesn't matter how fast they are going relative to the ground.

So, if you come up on a car at 30 m/s and it is only going 20 m/s it's relative velocity is -10 m/s. It is 20 m ahead, then pulls in front of you.

It is coming at you and you need negative acceleration to back off.

If you apply \( a(t) = -20 \text{ m/s}^2 \) what is the car's equation of motion?
First, use the ground's reference

\[ y(t) = y_0 + v_0 t + \frac{1}{2} a t^2 \]

- \( y_0 = 30 \text{ m/s} \)
- \( a = -20 \text{ m/s}^2 \)
- \( v_0 = 0 \)

\[ y = y_0 \]

\[ I(t) = I_0 + v_{0i} t + \frac{1}{2} a_i t^2 \]

- \( I_0 = 20 \text{ m/s} \)
- \( a_i = -20 \text{ m} \)
- \( v_{0i} = 0 \)

\[ I = I_0 + 20 \]

Diagram of a projectile's motion with coordinates and vectors.
What is the distance between the cars?

\[ D(t) = I(t) - y(t) = 20 + 20t - 30t + \frac{20}{2}t^2 \]

\[ = 20 - 10t + 10t^2 \]

\[ D(0) = 20, \quad D(1) = 20, \quad D(2) = 40 \]

Now, repeat it in your reference frame... where the origin is always on you.

\[ y(t) = 0 \]

\[ I(t) = I_0 + Vt + \frac{1}{2}At^2 \]

\[ = 20 - 10t + \frac{20}{2}t^2 \]

\[ = 20 - 10t + 10t^2 \]

Just like before.

What does the other guy see?

\[ J(t) = -20 + 10t - 10t^2 \]
This is just a change of coordinates, for position, velocity, and acceleration.

Just as with vectors, you simply subtract to find the coordinates in the new frame. We'll see this more in 2D & 3D next time.

**Variable Acceleration**

So far we've only considered \( a(t) = \text{constant} \).

If \( a(t) \) is a function of \( t \), then the same prescription will give us the equation of motion.

*E.g.* We can measure \( x(t) \) for a mass on a spring:

\[
\frac{a(t)}{\Lambda} = -\omega^2 A \cos \omega t
\]

\[
V(t) = -\omega A \sin \omega t + V_0
\]

\[
x(t) = A \cos \omega t + V_0 t + C
\]

\( C \) related to \( x_0 \)

actually \((x_0 - A)\)
Another example is changing your car’s acceleration as you let up on the gas.

\[ a(t) = 4 - 2t \quad \text{UNITS} \]

\[ v(t) = 4t - t^2 + V_0 \]

\[ x(t) = 2t^2 - \frac{1}{3}t^3 + V_0t + X_0 \]

Graph it:

Note: Negative acceleration

Beware ... this is non-constant \( a(t) \). Free-fall is constant \( a(t) \).
Sample problem.

Suppose you look at a baseball going up past your window and then back down. Does it take the same amount of time to cross the window going down as it did going up?

Do it by equations of motion and by graphing.

Suppose you measure $v(x)$ and find $v(x) \propto \sqrt{x}$. What can you say about the motion?

\[ V = \sqrt{x} \Rightarrow V^2 = x \]

\[ \frac{d}{dt} \text{ on both sides} \Rightarrow 2V \frac{dv}{dt} = \frac{dx}{dt} \Rightarrow 2v \ddot{a} = v \]

\[ \Rightarrow \ddot{a} = \text{constant.} \]

In fact if $a = \text{constant,}$

\[ V^2 = V_0^2 + 2a(x - x_0) \]

which is derived in the text and is often useful.