Last time we talked about vectors, which carry information about magnitude and direction.

The simplest case is 1-dimensional: \( +5 \text{ m} \)

\[ \Rightarrow \text{ magnitude} = 5 \text{ m} \]

\[ \text{Direction} = + \]

In two dimensions, we used coordinates to represent a vector:

\[ \vec{a} = (a_x, a_y) \]

or we used unit vectors.

\[ \vec{a} = a_x \hat{i} + a_y \hat{j} \]

In 3 dimensions:

\[ \vec{a} = (a_x, a_y, a_z) \quad \text{or} \quad \vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k} \]

We talked about adding, subtracting, and scaling vectors.

Geometric or Algebraic

\[ \implies \]

And we talked about multiplying vectors.
\[ \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = |\vec{A}| |\vec{B}| \cos \theta \]

Physically useful as component along a vector.

E.g., if \( \vec{B} = \hat{i} \), then \( \vec{A} \cdot \vec{B} = A_x \) etc.

We also talked about the cross product

\[ \vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k} \]

which gives a vector perpendicular to both \( \vec{A} \) & \( \vec{B} \).

Now 1-dimensional motion. Trying to start simple. Text is more rigorous. Read & Understand. I’ll try to make it intuitive. with an example.

A car race

\[ \]

Motion of cars is described by position at any time.
How do we describe the position? With a coordinate system.

I can arrange it anyway I want,

If it were a drag-race then $x=0$ at start and $t=0$ at start.

If it were the final straight-away in the Indy 500, then $x=0$ at the finish line, and maybe even $t=0$ at the end of the race.

Negative time? Sure why not, everything is a difference relative measurement anyway.

$t=0$ is not midnight or the beginning of the universe.

$v=0$ is relative to the track not globally e.g. including $v=1000$ MPH from rotation of the earth.

In fact, for the drivers, they are most concerned about their speed relative to the other cars.

So, let $x=0$ be the finish line and $t=0$ be when things get exciting... the final race to the finish.
Both cars are neck-\(\frac{1}{2}\)-neck (hood-n-hood ?)
\[x_1(0) = x_2(0) = -100 \text{ m (say for sake of an example)}.
\]

And both cars are going as fast as they can.

What does that mean? constant velocity.

Who wins... well whoever can (is) going faster.

\[x_1(t) = x_1(t=0) + v_1 t\]

\[x_2(t) = x_2(t=0) + v_2 t\]

\[\Rightarrow \text{This is just the definition} \]
\[v = \frac{\Delta x}{\Delta t}\]

Note that the units make sense.

\[x_1(t) = 0 \quad \text{when} \quad 0 = x_1(0) + v_1 t \quad t_1 = \frac{-x_1(0)}{v_1} = \frac{-(-100)}{v_1}\]

\[t_2 = \frac{-x_2(0)}{v_2} = \frac{-(-100)}{v_2}\]

Winner is guy that gets there first

\[\text{i.e. smallest time} = \text{largest velocity}\]

What about the minus signs? Just our definition.
We could have defined $x=0$ at "starting point."

\[ x_1(t) = 0 + v_1 t \quad \quad x_2(t) = 0 + v_2 t \]

Finish line is at $x = 100 \text{ m}$

100 = \frac{100}{v_1}

\[ t_1 = \frac{100}{v_1} \quad \quad t_2 = \frac{100}{v_2} \]

Same thing. I'll stick with finish line = 0.

This is pretty boring. Things are more interesting if, say, one car has a lead and the other wins by overtaking.

\[ x_1(t) = x_1(0) + v_1 t \quad \quad x_2(t) = x_2(0) + v_2 t \]

How much faster does I have to go to beat 2?

\[ x_1(t) = 0 = x_1(0) + v_1 t \]

\[ t_1 = -\frac{x_1(0)}{v_1} = t_2 = -\frac{x_2(0)}{v_2} \]

\[ v_1 = \frac{x_1(0)}{x_2(0)} v_2 \]

\[ \Rightarrow \text{If } x_2 \text{ "starts" half the distance } \]

\[ v_1 \text{ must be } > 2 v_2 \]
Useful to graph the motion, \( x(t) \)

\[ \text{[First draw a line...]} \]

\[
\begin{align*}
&\text{Constant velocity means straight, linear, } x(t). \\
&\text{This is still pretty boring... in a movie you know that is not the end -- too easy. Need to make velocity change.}
\end{align*}
\]

O.K. car 1 runs out of gas just after it passes car 2. What happens? It slows down.

Then what is \( x(t) \)?
The velocity decreases slowly. The slope of this curve is velocity, so it smoothly goes to horizontal.

We said \( V = \frac{\Delta x}{\Delta t} \), but that is the average velocity.

I could use average velocity:

At one time \( x_1(0) \) \( \neq \) \( x_2(0) \)
At finish \( x_1(t_f) \) \( \neq \) \( x_2(t_f) \) \( x_2 < x_1 \)

\[ \text{Average velocity is } \frac{x_1(t_f) - x_1(0)}{t_f - 0} \]

Not as exciting or complete. I can break it into pieces: with gas \( \neq \) without and find average \( V \).

Or break it into tiny time slices ... frame by frame then using average velocity matches pretty well.

Even better with a very fast camera.

That is the whole point of a derivative.

\[ \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \]
So, $V(t)$ is

![Graph showing $V_1(t)$ and $V_2(t)$ vs. time.]

Instantaneous v.s. average velocity is an important distinction.

Average velocity during race can be low since the race includes the pit stops, and milk drinking.

In fact, since the track is a circle, $\Delta x = 0$ over the course of the race, so the average velocity is 0!

How can that make any sense? Velocity is a vector, the magnitudes and directions add up to zero. "Going nowhere fast!"
Speed is a scalar, velocity is a vector. They are interchanged in common language, not in physics.

A speedometer measures speed. If you wanted velocity, you need to add a compass. That gives you instantaneous velocity of course.

Anyway, with a changing velocity, we can't describe the motion as

\[ x(t) = x(0) + v \cdot t \]

we have to include the fact that the velocity is changing with time. \( v \) is a function of \( t \).

Changing velocity is "acceleration". That word is mis-used in common language as well. E.g. do you accelerate when you stomp on the brake? Yes! Large negative acceleration.

"De-acceleration" is an attempt to include the vector nature of accel. We don't use it, rather \( a \) can be \( \mathbf{a} \).

Turning is also acceleration, even if the speed doesn't change, the direction does. Change in velocity vector.
Just like \( v = \frac{dx}{dt} \), \( a = \frac{dv}{dt} \)

Units of velocity are \( \frac{m}{s} \).

Units of acceleration are \( \frac{m}{s}/s = \frac{m}{s^2} \).

So, \( a = \frac{d^2x}{dt^2} = \ddot{x} \) and \( v = \dot{x} \) \( \Rightarrow \) shorthand.

So in the race to the finish, the guy who ran out of gas had negative acceleration.

\( \rightarrow \) period where \( v \) decreases linearly with time.

\( \rightarrow \) constant \( v \)

\( a = 0 \)

\( a = \text{constant} \)
What would mean if we had this?

He got rear-ended by another car then continually pushed, accelerated, by that car.

You should be able to think about motion graphically like this in addition to the equations.

Another example: A hockey puck bouncing off a wall

Visually

If you're on the other team.
if we put $x=0$ at the wall and $+x$ points to the left.
Perfectly valid as long as we're consistent.

What is $v(t)$? Shown below in red.

What is the acceleration? Shown in green.

No acceleration except the $S$-function at the wall.
Of course even ice has some friction so if we actually measured it, we'd probably see the puck slowing down.

Which (red or green) is what would happen due to friction?
A really need some parabolic curve.

So, if the measured position vs. time isn't straight in our hockey puck experiment, which is wrong, the measurement or the theory?

The theory is wrong. It was incomplete.

The experiment may be flawed if e.g. we used average velocity or it is not useful by insufficiently controlling friction.
Since we have the relations between $\ddot{x}$, $\dot{v}$, $\ddot{a}$, we can describe the motion with relatively simple math.

Given $x(t)$ e.g. from measurement,

$$v(t) = \dot{x}(t) = \frac{dx(t)}{dt}$$

$$a(t) = \ddot{x}(t) = \frac{d^2x(t)}{dt^2}$$

OR

Given $a(t)$, we can calculate $v(t)$ and $x(t)$.

This description of motion is called kinematics. The term "Dynamics" is used for describing what causes the acceleration. E.g. gravity.

Don't have to understand it to describe the motion it causes.

How do we go from $a(t)$ to $x(t)$?

In the car race $a(t) =$ ______

So

$$v(t) =$$

Like before, resort to small time intervals, $\Delta t$

$$v(t + \Delta t) = v(t) + a(t) \Delta t$$

$$v(t + 2\Delta t) = v(t) + a(t) \Delta t + a(t + \Delta t) \Delta t$$
Each of these $\Delta t$'s sums up.

$$V(t) = V(0) + \lim_{n \to \infty} \sum_{k=1}^{n} a(t_k) \Delta t_k$$

If we let $n \to \infty$, then the sum becomes an integral.

You may know, or be learning, this from calculus.

$$V(t) = \int a(t)$$

In fact, the "fundamental theorem of integral calculus" is

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a)$$

where $\frac{dF(x)}{dx} = f(x)$

To avoid any confusion from reusing symbols, I should say

$$\int_{a}^{b} f(t) \, dt = F(b) - F(a)$$

where $\frac{dF(t)}{dt} = f(t)$.

So, $\frac{dV(t)}{dt} = a(t)$, $\Rightarrow V(t_1) - V(t_0) = \int_{t_0}^{t_1} a(t) \, dt$

$$V(t_1) = V(t_0) + \int_{t_0}^{t_1} a(t) \, dt$$
Given \( a(t) \) and \( v \) at some time \( t_0 \), we can calculate \( v(t) \).

In the same way, \( \frac{dx(t)}{dt} = v(t) \)

So, \( x(t) = x(t_0) + \int_{t_0}^{t} v(t) \, dt \)

If we know \( a(t) \) for all \( t \), then we can predict \( x(t) \) for any \( t \). Frighteningly deterministic!

Don't worry about your future being too predictable. But, you can see why celestial motion is so easy to predict even without understanding why things move.

You have probably not covered integrals yet in M3B, so a lightening review. We saw the meaning above, of an integral. Calculating integrals is easy for the simple functions we'll consider here (and "impossible" for real problems).

\[
\int f(x) \, dx = F(x) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)
\]

Since \( \frac{d}{dx} \) of a constant is 0, we can always add an unknown constant, \( + C \).
So,

\[ \int A \, dx = Ax + C \]

\[ \int Ax \, dx = \frac{1}{2} Ax^2 + C \]

\[ \int A \cos Bx \, dx = \frac{A}{B} \sin Bx + C \]

Validate this by taking derivatives.

A simple (but common) example is constant acceleration.

\[ \alpha(t) = A \quad A \text{ can be } > 0 \text{ or } < 0, \quad \text{or } = 0! \]

\[ v(t) = \int \alpha(t) \, dt = \int A \, dt \]

\[ = At + C \]

What is \( C \)? Well \( v(0) = A \cdot 0 + C \) so \( C = \text{initial velocity}, v_0 \equiv v(t=0). \)

\[ x(t) = \int v(t) \, dt = \int (At + v_0) \, dt \]

\[ = \frac{1}{2} At^2 + V_0 t + C \]

\[ \boxed{x(t) = \frac{1}{2} At^2 + V_0 t + x_0} \quad \text{iff } \alpha(t) = A = \text{constant} \]
Sample Problem:

An astronaut space ship goes between two space ships 100 m apart.
She has constant $a = 5 \text{m/s}^2$ for 2 seconds.
1) What acceleration is needed to just stop?
2) How long does it take?

1) exactly opposite, $-5 \text{m/s}$ for 2 seconds

For 1st 2 seconds,

$x(t) = \frac{1}{2} \cdot 5 \cdot t^2$

$v(t) = at = 10 \text{ m/s} \ at \ t=2$

80 m to go at 10 m/s = 8 seconds.

Then slow down in the last 2 seconds.

If you've seen Mission to Mars you can see that the writer's didn't know kinematics.
Short bursts and a lot of coasting.

Draw $a(t)$, $v(t)$, $x(t)$
Some sample conceptual questions:

Describe the motion associated with each of the following graphs.

Which is most likely to be descriptive of each of the following:

(a) A ping-pong ball during a ping-pong game.
(b) Stop-and-go traffic
(c) The motion of a baton in a relay race.
Sample Conceptual Questions

1) In this graph:

   \[ X \quad t \]

   Is the object:
   (a) Always speeding up, \( a > 0 \)?
   (b) Always slowing down, \( a < 0 \)?
   (c) Both accelerating and "de-accelerating".

2) In this graph of two different objects' positions:

   \[ X \quad t \]

   When do they have the same velocity?
   When do they have the same acceleration?
   Never! \( a_1 > 0 \) always \( a_2 < 0 \) always.